M.S Comprehensive Exam
Theoretical Statistics Portion
Department of Statistics
January 11, 2001

INSTRUCTIONS

1. All problems are equally weighted.
2. Begin your solution to each problem on a separate sheet of paper. If you do not attempt a problem, turn in a blank page on which you have written the missing problem number and your identification number.
3. Be sure to hand in your solutions by arranging them in such a way that the problems 1-9 appear in order.
4. If you arrive at a conclusion which is obviously incorrect, indicate that you are aware that the conclusion is incorrect and elaborate, if possible.
5. If the answer to one part depends upon the results of earlier parts that you were not able to answer, demonstrate your competence on the remaining parts by making reasonable assumptions about answers to the missing parts.

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1. A merchant has found that the number of items of brand XYZ that he can sell in a day is a Poisson random variable with mean 1.

   (a) How many items of brand XYZ should the merchant stock to be 95 percent certain that he will have enough to last for 2 days? (Give a numerical answer without normal approximation.)

   (b) How many items of brand XYZ should the merchant stock to be 95 percent certain that he will have enough to last for 100 days? (Give a numerical answer using normal approximation. Hint: If $Z \sim N(0, 1)$, then $P(Z < 1.645) = 0.95$).

   (c) What is the expected number of days out of 100 that the merchant will sell no items of brand XYZ?

2. Let random variables X and Y have finite first and second moments.

   (a) Prove that

$$(E[XY])^2 \leq E(X^2)E(Y^2)$$

   with equality if and only if $P[Y = cX]$ for some constant c.

   (b) Prove that

$$\mathrm{Var}(X+Y) \leq \left(\sqrt{\mathrm{Var}(X)} + \sqrt{\mathrm{Var}(Y)}\right)^2.$$

3. Suppose that X_1, X_2 are iid random variables with the density function

$$f(x) = \frac{1}{x^2} I_{(1,\infty)}(x).$$

Suppose that

$$Y_1 = \frac{X_1}{X_1 + X_2}$$

and

$$Y_2 = X_1 + X_2.$$

(a) Find the joint density function of Y_1 and Y_2.

(b) Find the marginal density functions of Y_1 and Y_2.

(c) Find $E\left(Y_2 | Y_1 = \frac{1}{4}\right)$.

4. Let X_1, X_2 be a random sample of size 2 from the density function.

$$f(x;\theta) = \begin{cases} \frac{1}{\theta} 2x e^{-x^2/\theta}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the UMVUE of θ.

(b) Let $W_1 = X_1/X_2$ and $W_2 = X_1^2 + X_2^2$. Use sufficiency and completeness arguments to show that W_1 and W_2 are independent.

5. Let $X_1, \cdots, X_n$ be a random sample from a $N(\mu, \sigma^2)$ distribution. Define

$$U = \frac{1}{n}\sqrt{\frac{\pi}{2}} \sum_{k=1}^{n} |X_k - \mu|$$

assuming μ is known.

(a) Show that U is an unbiased estimator of σ.

(b) Find the efficiency of U.

6. Suppose an observable random sample, $X_1, \cdots, X_n$, comes from the probability density function

$$\begin{aligned} f(x;\theta) &= e^{-(x-\theta)}, \ \ x > \theta \\ &= 0, \ \text{elsewhere} \end{aligned}$$

(a) Construct a confidence interval for θ based on the sufficient statistic for θ. Use $(1-\alpha)$ as the confidence coefficient.

(b) Construct a confidence interval for θ based on $\overline{X}$. Use $(1-\alpha)$ as the confidence coefficient.

(c) Which interval do you prefer? Why?

7. Suppose that $X_1, X_2, \cdots, X_n$ are iid random variables with a beta$(\mu, 1)$ pdf and that $Y_1, Y_2, \cdots, Y_m$ are iid with a beta$(\theta, 1)$ pdf. Assume further that the X's are independent of the Y's.

   (a) Derive the LRT for $H_0 : \theta = \mu$ versus $H_a : \theta \neq \mu$. Show that the LRT can be based on the statistic

$$T = \frac{\Sigma 1 n X_i}{\Sigma 1 n X_i + \Sigma 1 n Y_j}.$$

   (b) Find the distribution of T when H_0 is true and then show how to get of test of size $\alpha = .10$.

8. Suppose that $X_1, X_2, \cdots, X_n$ be a random sample from

$$f(x : \theta) = (1/\theta)x^{(1-\theta)/\theta} I_{(0,1)}(x)$$

(a) Show that this family of distributions has monotone likelihood ratio in some statistic T.

(b) If we are interested in testing

$$H_0 : \theta \leq \theta_0 \text{ versus } H_1 : \theta > \theta_0$$

derive the form of the *uniformly most powerful* α-level test. Give reasons for any claims that you make.

(c) If $X_1, X_2, \cdots, X_n$ have the distribution given above, then $-1n(X_i)$ has an exponential distribution with mean θ (parameter $\lambda = 1/\theta$). Derive an expression for the value of the power function $\pi(\theta)$, that can be evaluated by using standard statistical tables.

(d) If someone approached you with another test for the same hypotheses and this new test has size $\leq \alpha$, what can you say about the value of the power function of this new test at $\theta = \theta_1 > \theta_0$?

9. Let $X_1, X_2, \cdots, X_n$ be a random sample from a normal population with unit variance and consider the test that rejects $H_0 : \mu = 0$ in favor of $H_a : \mu > 0$ if $\overline{X}_n > k_n$, with k_n defined so that the test has size α.

   (a) Give an expression for k_n that can be evaluated by using standard statistical tables.

   (b) Derive an expression for the power function of the test considered above.

   (c) Show that for each fixed n, that above test is *unbiased.*

   (d) Show that the above sequence of tests is *consistent.*