

M.S Comprehensive Exam
Theoretical Statistics Portion
Department of Statistics
August 21, 2001
9:00 A.M. - 1:00 P.M.

INSTRUCTIONS

1. All problems are equally weighted.
 2. Begin your solution to each problem on a new sheet of paper. If you do not attempt a problem, turn in a blank page on which you have written the missing problem number and your identification number.
 3. Be sure to hand in your solutions by arranging them in such a way that the problems 1-9 appear in order.
 4. If you arrive at a conclusion which is obviously incorrect, indicate that you are aware that the conclusion is incorrect and elaborate, if possible.
 5. If the answer to one part depends upon the results of earlier parts that you were not able to answer, demonstrate your competence on the remaining parts by making reasonable assumptions about answers to the missing parts.
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1. Let $X_0, X_1, X_2, \dots, X_n$ be identically distributed $N(0, 1)$ random variables and Y_1, Y_2, \dots, Y_n be identically distributed $U(0, 1)$ random variables where the X 's and Y 's are mutually independent. Define

$$W_i = Y_i X_i + (1 - Y_i) X_0 \quad i = 1, 2, \dots, n \quad \text{and} \quad S = \sum_{i=1}^n W_i.$$

- (a) Find the mean and variance of S .
 (b) Find a value of t so that $P(S^2 \geq t) \leq .05$.
2. Suppose that $Y_1 < Y_2 < Y_3$ is the order statistic corresponding to a random sample of size 3 from a distribution with density function

$$f(x) = \begin{cases} kx, & 0 \leq x < M \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Determine k (as a function of M) so that the above function is a bona fide density function.
 (b) Let

$$Z_1 = \frac{Y_1}{Y_2}, \quad Z_2 = \frac{Y_2}{Y_3}, \quad \text{and} \quad Z_3 = Y_3.$$

Show that Z_1, Z_2 and Z_3 are mutually independent random variables.

- (c) Show that $E(Z_2) = E(Y_2/Y_3)$ does not depend on M .
3. Suppose that B is a random variable with cumulative distribution function

$$F_B(b) = \begin{cases} e^{-b^{-1}}, & 0 \leq b < \infty \\ 0, & b \leq 0. \end{cases}$$

Given that $B = b$, X has an exponential density function with density

$$f(X|b) = \frac{1}{b} e^{-x/b} \times I_{(0, \infty)}(x)$$

- (a) Give the joint density of (X, B) .
 (b) Compute $P(1.5 < X < 3|B = 1)$.
 (c) Find the marginal function of X .

4. Let X_1, \dots, X_n denote a random sample from the probability density function.

$$\begin{aligned} f(x; \theta) &= \theta e^{-\theta x}, \quad x > 0, \\ &= 0, \quad \text{elsewhere} \end{aligned}$$

- Find the *MLE* of θ .
- Is the *MLE* unbiased for θ ? If not, adjust it to be unbiased.
- Is the adjusted *MLE* from part (b) consistent for θ ?
- Find the variance of the adjusted *MLE* from part (b).
- The ratio of the Cramér-Rao lower bound to the variance of an unbiased estimator, T , is called the efficiency of T . Find the efficiency of the adjusted *MLE* for estimating θ .
- For what function of θ does a fully efficient (efficiency = 1) estimator exist?
- It is desired to estimate

$$P(X > t) = e^{-\theta t}$$

Find an estimator of $e^{-\theta t}$ and a standardization of that estimator that converges in distribution to a standard normal random variable.

5. Let X_1, \dots, X_n denote a random sample from

$$f(x; \theta) = \begin{cases} \frac{1}{\theta} r x^{r-1} e^{-x^r/\theta}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

- Find an estimator of θ by the method of moments.
 - Find the *MLE* of θ .
 - Are the estimators in (a) and (b) consistent? (Show why or why not.)
 - Which estimator, (a) or (b), would you favor using and why?
 - Based on the *MLE* for θ , find an unbiased estimator of $E(X_i)$.
6. Let Y_1, Y_2, \dots, Y_n be a random sample from the probability density function given by

$$\begin{aligned} f(y) &= k\beta^k y^{-(k+1)}, \quad y > \beta \\ &= 0, \quad \text{elsewhere} \end{aligned}$$

where k is assumed known and $k, \beta > 0$. Find a confidence interval for β with confidence level γ .

- State and prove the Neyman-Pearson Lemma.
- Let X_1, \dots, X_n be a random sample from the density function

$$f(x, \theta) = \frac{2x}{\theta^2} e^{-x^2/\theta^2} I_{(0, \infty)}(x).$$

Derive the most powerful test of $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1 (\theta_0 < \theta_1)$.

- Obtain an α -level test in (b) with a familiar distribution.
- How do you compute the power of the test in (c)?

8. Let X_1, \dots, X_n be a random sample from a Weibull distribution with density

$$f(x; \lambda) = \lambda c x^{c-1} e^{-\lambda x^c},$$

where c is a known positive constant and $\lambda > 0$ is the parameter of interest.

- (a) Does the uniformly most powerful test for testing $H_0 : \frac{1}{\lambda} \leq \frac{1}{\lambda_0}$ vs. $H_1 : \frac{1}{\lambda} > \frac{1}{\lambda_0}$ exist? Why?
 - (b) If your answer is yes in (a), find the critical value of an α -level test in (a) with a familiar distribution.
 - (c) Suppose $\frac{1}{\lambda_0} = 12$. Find the sample size needed for a level 0.01 test to have power at least 0.95 at the alternative value $\frac{1}{\lambda_1} = 15$. Use the normal approximation to the critical value and the probability of rejection. ($z_{0.01} = 2.33$, $z_{0.05} = 1.645$).
9. Let X_1, X_2, \dots, X_n be a random sample from a $N(\mu, \sigma^2)$ distribution, where μ and σ^2 both are unknown. Derive the likelihood ratio test of $H_0 : \mu = 0$ vs. $\mu \neq 0$.