1. Suppose that $A \in \mathbb{C}^{n \times n}$ is Hermitian with $\lambda_1$ and $\lambda_n$ the smallest and largest eigenvalues of $A$ respectively. Show that

$$\lambda_1 \|x\|^2 \leq x^* Ax \leq \lambda_n \|x\|^2$$

for all $x \in \mathbb{C}^n$.

2. (a) Suppose $p$ and $q \in \mathbb{R}$ with $p$ and $q$ positive and $p^{-1} + q^{-1} = 1$. Show that for any matrix $A \in \mathbb{C}^{m \times n}$, we have $\|A\|_p = \|A^*\|_q$, where $A^*$ is the conjugate transpose of $A$.

(b) Prove that

$$\|A\|_2^2 \leq \|A\|_p \|A\|_q$$

for any $A \in \mathbb{C}^{m \times n}$ and any positive $p$ and $q \in \mathbb{R}$ with $p^{-1} + q^{-1} = 1$.

(c) Prove that for any $p \geq 1$ and any diagonal matrix $D \in \mathbb{C}^{n \times n}$, we have

$$\|D\|_p = \max\{|d_{ii}| : 1 \leq i \leq n\}.$$

3. (a) Show that the eigenvalues of a projector are either 0 or 1.

(b) Show that a projector $P$ is orthogonal if and only if $P = P^*$.

4. Show that the element of largest magnitude in a Hermitian, positive definite matrix lies on the diagonal.

5. Suppose that $A \in \mathbb{C}^{n \times n}$ is an invertible matrix, $u \in \mathbb{C}^n$, and $v \in \mathbb{C}^n$.

(a) Show that

$$\det(A - uv^*) = (1 - v^*A^{-1}u) \det A.$$

(b) Suppose that $A$ is a real diagonal matrix and $u = v$ with $u_i \neq 0$ for each $i$. Show that the eigenvalues of $A - uu^*$ are the roots of

$$\sum_{i=1}^n \frac{|u_i|^2}{\alpha_{ii} - \lambda} = 1.$$