1. Let \( (X_m)_{m=1}^\infty \) be an i.i.d. sequence of real valued variables on \((\Omega, \mathcal{F}, P)\). Let \( \mathcal{F}_m = \sigma(X_1, \ldots, X_m) \). Put \( S_n = X_1 + \cdots + X_n \), for \( n \geq 1 \) and put \( S_0 = 0 \).

Which of the following sequences are martingales relative to \( \mathcal{F}_m \). For those which are not martingales w.r.t. this filtration, what condition have to be
a) \( S_m, n = 0, 1, 2, \ldots, \quad E[|X_1|] < \infty \)

b) \( X_1^2 + \cdots + X_m^2 = \lambda - \lambda I, \quad m = 1, 2, \ldots, \quad E(X_i^2) < \infty \)

\( \lambda \) real

c) \( \exp (\alpha S_m - \lambda) = 0, \quad m = 0, 1, 2, \ldots \) where

\( \Phi(\alpha) = E[\exp \alpha X_i] < \infty, \quad \alpha \) real

d) \( Y_m = S_{\min}(n, T) \)

Where

\( P(X_1 = \pm 1) = \frac{1}{2} \) and \( T = \min(n > 0, S_m = 0) \)

Let \( \{\alpha_m\}_{m=1}^\infty \) be a sequence of probability measures on \( \mathbb{R} \). Show that the following are equivalent:

a) There exists a probability measure \( \alpha \) on \( \mathbb{R} \) such that

\[ \lim_{m \to \infty} \alpha_m(\{T\}) = \alpha(\{T\}) \]
3. For all closed intervals $I$ on $\mathbb{R}$ whose end points are continuity points of $\alpha$.

b) If $\{F_m\}_{n=1}^{\infty}$, respectively $F$ are the distribution functions of $\xi_m$, respectively $\xi$, 

\[
\lim F_m(x) = F(x)
\]

at every point $x$ of continuity of $F$.

3. Let $\{\xi_m\}_{n=1}^{\infty}$ be a sequence of probability measures on $\mathbb{R}$ and let $\{\phi_m(t)\}_{n=1}^{\infty}$ be the corresponding characteristic functions. Assume that $\lim_{n} \phi_m(t) = \phi(t)$ holds.

If $t \in \mathbb{R}$ and $\phi(t)$ is continuous at $t = 0$.

Note that $\phi(t)$ is the characteristic function of some probability measure and that the conditions of Problem 12 hold.
4. If $X$ is a real-valued random variable, show that for $0 < \varepsilon \leq 1$,

$$E\left[ \frac{|X|}{1 + |X|} \right] \leq \varepsilon + P\left[ |X| > \varepsilon \right]$$

and that

$$P\left( |X| > \varepsilon \right) \leq \frac{1 + E\left[ \frac{|X|}{1 + |X|} \right]}{\varepsilon}$$

Define that the set of real-valued random variables on a given probability space can (with suitable identification) be regarded as a metric space if the distance between two random variables $X$ and $Y$ is defined to be

$$d(X, Y) = E\left[ \frac{|X - Y|}{1 + |X - Y|} \right]$$

and that convergence in this metric space then corresponds to convergence in probability.
[5a] Let \( X_n, n = 1, 2, \ldots \) be i.i.d. \( \mathbb{S} \),

\[ E[X_n] = 1 \] and \( \bar{Y}_n = \frac{1}{n} \sum_{k=1}^{n} X_k \).

Prove that \( \bar{Y}_n \) is a martingale relative to the
filtration \( \mathcal{F}_n = \sigma(\{X_1, \ldots, X_n\}) \).

Show that \( \lim_{n \to \infty} \bar{Y}_n = \bar{Y} \) almost surely.

b) Show that

\[ P[\bar{Y}_\infty = 0] = 0 \text{ or } 1. \]

Hint: Recall infinite products and the zero-one law.

c) (Using b)) Show that

\[ E[\bar{Y}_\infty] = 1 \iff P[\bar{Y}_\infty > 0] = 1. \]

[6] Let \( X_n, n = 1, 2, \ldots \) be a supermartingale
such that \( E[|X_{n+1} - X_n| \mid \mathcal{F}_n] \leq M \) a.s.

Show that for any two optional times \( S \leq T \),

\[ E[T \mid \mathcal{F}_S] \leq T \text{, we have } E[|X_T|] < \infty \text{ and } \]

\[ E[|X_T - X_S| \mid \mathcal{F}_S] \leq E[X]. \]