Ph.D. Qualifying Exam in Probability
May 1989

1. Let $X_n$ be independent identically distributed random variables and $S_n = \sum_{1}^{n} X_1$.

Suppose $E[\sup_{n} |\frac{S_n}{n}|] < \infty$.

Show that

$E[|X_1| \log^+ |X_1|] < \infty$.

Where $\log^+ x = \log x$ if $x \geq 1$

$= 0$ otherwise.

Hint: $C_n = \prod_{j=1}^{n} P[|X_j| \leq j] + c > 0$.

If $T = \inf \{n: |X_n| > n\}$ then

$\sum_{1}^{\infty} \frac{1}{n} \int_{(T-n)}^{\infty} |X_n| dp = \sum_{1}^{\infty} \frac{C_{n-1}}{n} \int_{1}^{\infty} |X_1| dp$

$\sum_{1}^{\infty} \frac{1}{n} \sum_{(T-n)}^{\infty} |S_{n-1}| dp < \infty$.

2. Let $X_n$ be independent Poisson variables. Suppose $\sum_{1}^{\infty} X_n < \infty$ a.s.

Show that $\sum_{1}^{\infty} E[X_n] < \infty$.

Hint. Borel Cantelli.

3. a) Define the notion of conditional expectation. In particular explain the notion $E[X|Y]$ where $X$ and $Y$ are random variables.
b) Suppose X, Y are random variables such that \(E(X^2),\)
\(E(Y^2) < \infty\)
and \(E[X|Y] = Y\)
\(E[Y|X] = X\)
Show that \(X = Y\) a.s.

4. a) A sequence of characteristic functions on \(\mathbb{R}^1\) converging
pointwise to a characteristic function does so uniformly on
compact sets. Explain why - no need to give complete proofs.

b) Let \(b_n \in \mathbb{R}\) and \(f\) a characteristic function not identically
equal to \(1\). Suppose \(f(b_n t)\) converges to a characteristic
function. Show that \(\lim b_n = b \in \mathbb{R}\).

c) Let \(\phi(t) = \frac{1}{8}[1 + 7e^{it}]\). Show that \(\phi\) is a characteristic
function but that \(|\phi|\) is not.

\textbf{Hint.} \(|\phi|^2\) is the characteristic function of a measure
concentrated on \([-1,0,1]\).

If \(|\phi|\) is the characteristic function of a measure \(m\).
Then \(|\phi|^2\) is the characteristic function of \(m^*m\).

5. Let \(X_t\) be a standard Brownian motion.

(i) Show that \((X_t), (X_t^2 - t)\) and \(e^{uX_t - \frac{1}{2}u^2t}\) are all
martingales.
(ii) Let \( T_a(w) = \inf \{ t : X_t(w) = a \} \) for \( a > 0 \). The inf being defined to be \( +\infty \) if the set is empty. Using (1) Compute \( E[e^{-\lambda T_a}] \) for \( \lambda > 0 \), and show that \( T_a < +\infty \) a.s. and \( E[T_a] = +\infty \).

6. State and sketch the proof of the martingale convergence theorem.

7. Using the martingale convergence theorem prove the following:

Let \( F_n \) be an increasing sequence of \( \sigma \)-fields, \( X_n \) a sequence of random variables dominated by an integrable random variable \( Y: |X_n| \leq Y \). Suppose \( X_n \rightarrow X \) a.s.

Then \( E[X_n|F_m] \rightarrow E[X|F_\infty] \) a.s.

**Hint.** Put the \( U_m = \sup_{m \geq n} |X_m - X| \)

Then for \( m > n \)

\[
E[|X_m - X| | F_m] \leq E[U_n | F_m].
\]

\[
E(\lim_{m \to \infty} E[|X_m - X| | F_m]) \leq E[E[U_n | F_\infty]] = E[U_m]
\]