1. Let $U$ be either $\mathbb{R}_+^n$ or the unit ball in $\mathbb{R}^n$ ($n \geq 2$). Write the expression of the Green’s function for $-\Delta$ with respect to the Dirichlet condition on $\partial U$.

2. Let $U \subset \mathbb{R}^n$ be a bounded open set with smooth boundary $\partial U$, $A = (a_{ij})_{n \times n}$ be a symmetric, positive definite constant matrix. For the following variational form:

$$I(w) := \frac{1}{2} \int_U \sum_{i,j=1}^n a_{ij} \partial_i w \partial_j w \, dx; \quad w \in H^1_U := \{ w \in H^1(U); \quad w = g \text{ on } \partial U \}$$

where $g$ is a smooth function on $\partial U$, prove that the minimizer $u$ exists and is smooth.

3. Let $A = (a_{ij})_{n \times n}$ be a symmetric and positive definite constant matrix. Suppose $u$ satisfies

$$\sum_{i,j=1}^n a_{ij} \partial_i u = 0, \quad \text{in } \mathbb{R}^n$$

and $u \geq -1$ in $\mathbb{R}^n$. Show that $u \equiv \text{constant}$.

4. Let $\phi(x) = \frac{1}{\alpha(n)} |x|^{2-n}$ be the fundamental solution for the Laplace operator in $\mathbb{R}^n$ ($n \geq 3$). Here $\alpha(n)$ is the volume of the unit ball $B_1$. Suppose $f(x) \in C^2(\mathbb{R}^n)$ and satisfies

$$|D^j f(x)| \leq C(1+|x|)^{-2-\epsilon-j}, \quad \forall x \in \mathbb{R}^n, \quad j = 0, 1, 2$$

where $\epsilon$ is a positive number. Then

$$u(x) = \int_{\mathbb{R}^n} \phi(x-y)f(y)dy$$

satisfies

$$-\Delta u(x) = f(x) \quad \mathbb{R}^n.$$
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6. Write down an explicit formula for a solution of
\[
\begin{cases}
  u_t - \Delta u + cu = f, & \mathbb{R}^n \times (0, \infty), \\
  u = g & \mathbb{R}^n \times \{t = 0\}
\end{cases}
\]
where $c \in \mathbb{R}$.

7. Let $U \subset \mathbb{R}^n$ be an open, bounded subset of $\mathbb{R}^n$ with smooth boundary. Set $U_T = U \times [0, T]$, $\Gamma_T = \overline{U_T} \setminus U_T$ where $T > 0$. Prove that there exists at most one solution $u \in C^2(U_T)$ of
\[
\begin{cases}
  u_{tt} - \Delta u = f, & U_T, \\
  u = 0 & \Gamma_T \\
  u_t = h, & U \times \{t = 0\}
\end{cases}
\]
where $g, h$ are smooth functions.