University of Florida  (2020)  
Ph. D. Examination in Partial Differential Equations

Instructions. Do all problems 1–4. Choose one problem of 5 and 5' and one problem of 6 and 6'.

(1) Let $B$ be the unit ball in $\mathbb{R}^3$. For which values of $\alpha$ does the function $|x|^\alpha$ belong to $H^1(B)$? Justify your answer.

(2) Let $U \subset \mathbb{R}^n$ be open and bounded. Let $u_m$ and $v_m$ be bounded sequences in $H^1(U)$. Show that there exist subsequences $u_{m_j}$ of $u_m$ and $v_{m_j}$ of $v_m$, and functions $u, v \in H^1(U)$, such that

$$
\int_U u_{m_j} v_{m_j} \to \int_U uv,
$$

and

$$
\int_U D_{x_i}(u_{m_j} v_{m_j}) \to D_{x_i}(uv), \quad i = 1, \ldots, n,
$$

as $j \to \infty$.

(3) Let $U \subset \mathbb{R}^n$ be open, bounded, and connected with smooth boundary $\partial U$. Let $u(x,t)$ be a smooth solution to the IBV problem:

$$
u_t = \Delta u, \quad (x,t) \in U_T,$$

$$
\frac{\partial u}{\partial n} = 0, \quad (x,t) \in \partial U \times [0,T],
$$

$$
u(x,0) = f(x), \quad x \in U.
$$

Let

$$
(u)_U(t) = \frac{1}{|U|} \int_U u(x,t) \, dx.
$$

Show that

(a) $\frac{d}{dt} (u)_U(t) \equiv 0$.
(b) $u \to (u)_U$ in $L^2(U)$, as $t \to \infty$.

Hint: Consider the equation for $v = u - (u)_U$, and use energy estimate for the equation of $v$, and the Poincare’s inequality.
(4) Let $U \subset \mathbb{R}^n$ be open, bounded, and connected with smooth boundary $\partial U$. Let $L$ be a uniformly elliptic differential operator of the form:

$$Lu = -\sum_{i,j=1}^{n} a_{ij}u_{x_i x_j},$$

Suppose that $f$ is a bounded function and $u, v \in C^2(\overline{U})$ satisfy

$$Lu = f, \quad Lv \geq 1, \quad x \in U,$$

$$u(x) = 0, \quad v(x) \geq 0, \quad x \in \partial U.$$  

Suppose that there exists a point $x_0 \in \partial U$, such that $v(x_0) = 0$. Prove that there exists a constant $C > 0$, such that

$$|Du(x_0)| \leq C|\frac{\partial v}{\partial n}(x_0)|.$$

(5) Let $U \subset \mathbb{R}^n$ be open and bounded with smooth boundary $\partial U$. We say that a function $u \in H^2_0(U)$ is a weak solution of the biharmonic equation:

$$\Delta^2 u = f, \quad x \in U, \quad (5.1)$$

$$u = 0, \quad \frac{\partial u}{\partial n} = 0, \quad x \in \partial U, \quad (5.2)$$

if

$$\int_{U} \Delta u \Delta v \, dx = \int_{U} fv \, dx, \quad \text{for any } v \in H^2_0(U). \quad (5.3)$$

Prove the following statements:

(a) For a given $f \in L^2(U)$ there is an unique solution to the problem (5.3).

(b) If $u \in C^4(\overline{U})$ and $f \in C^0(\overline{U})$ satisfy (5.3), then $u$ satisfies (5.1) at each point $x \in U$. 

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(5') Let $U \subset \mathbb{R}^n$ be open and bounded. Show that $u \in H^1_0(U)$ is a weak solution to the boundary value problem

$$-\Delta u + u = 1, \quad x \in U,$$

$$u = 0, \quad x \in \partial U,$$

if only if $u$ minimizes

$$E(v) = \int_U (|\nabla v|^2 + v^2 - 2v) \, dx$$

over $v \in H^1_0(U)$.

(6) Let $U \subset \mathbb{R}^n$ be open and bounded. We say $v \in C^2(\overline{U})$ is subharmonic if $-\Delta v \leq 0$ for all $x \in U$.

(a) Prove that for any subharmonic $v$,

$$v(x) \leq \int_{B(x,r)} v(y) \, dy, \quad \text{for all } B(x,r) \subset U.$$

(b) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Suppose that $u$ is harmonic and $v = \phi(u)$. Prove that $v$ is subharmonic.

(6') Let $U \subset \mathbb{R}^n$ be open and bounded with smooth boundary $\partial U$.

(a) Write the fundamental solution $\Phi(x)$ of Laplace’s equation.

(b) Prove that for any point $x \in U$ and any function $u \in C^2(\overline{U})$,

$$u(x) = \int_U \Phi(y - x) \Delta u(y) \, dy$$

$$+ \int_{\partial U} \Phi(y - x) \frac{\partial u}{\partial n}(y) - u(y) \frac{\partial \Phi}{\partial n}(y - x) \, dS(y),$$

where $\Phi$ is the fundamental solution of Laplace’s equation, and $n$ is the outward unit normal to $\partial U$. 

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