Directions: You have 2 hours to answer the following questions. You must show all your work as neatly and clearly as possible and indicate the final answer clearly. You may not use a calculator.

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(1) (20 points) Show that the fixed point equation \( x = f(x) \) with a fixed point \( \alpha \) can be solved iteratively, if \( f'(\alpha) \neq 1 \), by one of the following fixed point iteration formulas:

(a) \( x_{n+1} = f(x_n) \)

(b) \( x_{n+1} = f^{-1}(x_n) \).

Hint: The following formula for the derivative of an inverse is valid

\[
[f^{-1}]'(x) = \frac{1}{f'(f^{-1}(x))}
\]

(c) \( x_{n+1} = (x_n + f(x_n))/2 \)
(2) (20 points) Let \( f(x) \in C[a, b] \). Let \( p(x) \) be a polynomial for which
\[
\|f' - p\|_\infty \leq \epsilon
\]
and define
\[
q(x) = f(a) + \int_a^x p(t)dt, \quad a \leq x \leq b.
\]
Show that \( q(x) \) is a polynomial that satisfies
\[
\|f - q\|_\infty \leq \epsilon(b - a)
\]
(3) (20 points) Consider the integral

$$\int_0^\pi x^2 \cos x \, dx$$

(a) Consider a quadrature rule of the form

$$\int_0^\pi x^\alpha f(x) \, dx \approx Af(0) + B \int_0^\pi f(x) \, dx$$

where $\alpha > -1, \alpha \neq 0$ is a parameter. Determine the constants $A, B$ so that the quadrature formula has degree of exactness one.

(b) Use the formula in part (a) to approximate the integral (1).
(4) (20 points) For the function $f(x) = \ln(1 + x)$ for $x \in [0, 1]$, find the minimax approximation polynomial of degree one. Give the exact value of the minimax error.
(5) (20 points) Assume that you are solving the initial value problem
\[
y' = f(t, y) \quad a \leq t \leq b \\
y(a) = \alpha
\]

(a) Derive the formula for the global error of the numerical solutions for the ODE problem above obtained via Euler’s method.
Hint: The formula is \( M = ||Y''||_\infty \):
\[
|Y(t_i) - w_i| < \frac{hM}{2L} [e^{L(b-a)} - 1].
\]

(b) Compute the value of \( M = ||Y''||_\infty \) necessary to apply the global error formula above to the specific ODE problem
\[
y' = \sin(t + 2y) + e^t \quad 0 \leq t \leq 1 \\
y(0) = 0
\]