Numerical Analysis Qualifying Exam (Spring 2010). 

1. Prove that a projector is normal if and only if it is self-adjoint.

2. Let $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$. Consider the following iteration for solving $Ax = b$, that computes $x_{n+1}$, given $x_n \in \mathbb{R}^n$, as follows (in $m$ intermediate steps): Setting $x_{n+1}^{(0)} = x_n$, for $\ell = 1, 2, \ldots, m$, compute $x_{n+1}^{(\ell)} = x_{n+1}^{(\ell-1)} + \tau_\ell (b - Ax_{n+1}^{(\ell-1)})$. Then, define $x_{n+1} = x_{n+1}^{(m)}$. There is a linear operator $E$ such that $x_{n+1} - x = E(x_n - x)$. Give a formula for $E$. Suppose $A$ is Hermitian and positive definite with spectral condition number $\kappa$. Prove that there are real values of the $m$ parameters $\tau_\ell$ such that 

$$
\rho(E) \leq 2 \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^m.
$$

3. Suppose $A$ and $B$ are square matrices such that $AB$ is normal. Prove that $\|AB\|_2 \leq \|BA\|_2$. (We use $\| \cdot \|_2$ to denote the spectral norm and $\| \cdot \|$ to denote any induced matrix norm.)

4. Let $A \in \mathbb{C}^{m \times m}$, and $a_j$ be its $j$-th column. Prove that $|\det A| \leq \prod_{j=1}^m \|a_j\|_2$.

5. Suppose $A$ is a Hermitian positive definite matrix split into $A = C + C^* + D$ where $D$ is also Hermitian positive definite. Show that $B = C + \omega^{-1}D$ is invertible. Consider the iteration $x_{n+1} = x_n + B^{-1}(b - Ax_n)$, with any initial iterate $x_0$. Prove that $x_n$ converges to $x = A^{-1}b$ whenever $0 < \omega < 2$.

6. Let $x_0, x_1, \ldots, x_n$ be distinct points in a finite interval $[a, b]$ and $f \in C^1[a, b]$. Show that for any given $\varepsilon > 0$ there exists a polynomial $p$ such that $\|f - p\|_\infty < \varepsilon$ and $p(x_i) = f(x_i)$ for all $i = 0, 1, \cdots, n$ (where $\| \cdot \|_\infty$ denotes the $L^\infty(a, b)$-norm).

7. Let $p > 0$ and $x = \sqrt{p + \sqrt{p + \sqrt{p + \cdots}}}$, where all the square roots are positive. Design a fixed point iteration $x_{n+1} = F(x_n)$ that converges to $x$. You should write down a specific $F$. Obtain a sufficient condition on the initial iterates for (global) convergence of the iteration.

8. Let $n$ be a positive integer, $h = 1/n$, and consider the grid of points $(ih, jh)$ for $i, j = 0, 1, \ldots, n$. Let $A$ be the finite difference operator with the “5-point stencil” discretizing the Laplace operator $-\Delta$, with zero Dirichlet boundary conditions on this grid. Describe it. Prove that the spectrum of $A$ consists of the numbers

$$
\lambda_{lm} = 4h^{-2}(\sin^2(l\pi h/2) + \sin^2(m\pi h/2)), 
$$

for all $l, m = 1, \ldots, n - 1$.

9. Let $x_m$ and $x_{m+1}$ be two successive (complex) iterates when Newton’s method is applied to a polynomial $p(z)$ of degree $n$. Prove that there is a zero of $p(z)$ in the disk $\{z \in \mathbb{C} : |z - x_m| \leq n|x_{m+1} - x_m|\}$.

10. Let $w(x)$ be a positive integrable function on $[a, b]$. Consider the quadrature

$$
\int_a^b f(x) w(x) \, dx \approx \sum_{k=0}^n (A_k f(x_k) + B_k f'(x_k) + C_k f''(x_k))
$$

for any $f$ with continuous first and second derivatives ($f'$ and $f''$, respectively). Give conditions on $A_k, B_k, C_k$ and $x_k$ so that the quadrature has precision $4n + 3$. 

Answer any 8 questions.