Numerical Analysis
Preliminary Exam
May 17, 2001

Do any 8 of the following 10 problems.

Part 1: Numerical Linear Algebra

1. (a) Under what conditions does a real \( n \times n \) matrix \( A \) have a Schur decomposition \( A = U^T T U \), where \( U \) and \( T \) are real \( n \times n \) matrices with \( U \) orthogonal and \( T \) upper triangular?
(b) Under what conditions on \( A \) is \( T \) diagonal?
(c) Does the matrix

\[
A = \begin{pmatrix}
1 & 0 & 1 \\
1 & -1 & 0 \\
0 & 1 & 1
\end{pmatrix}
\]

have an orthogonal set of eigenvectors?
(d) Give a careful statement of the singular value decomposition of a real \( m \times n \) matrix.
(e) Under what conditions does the singular value decomposition exist?

2. Consider the linear system \( Ax = b \).
(a) If the rows of \( A \) are linearly independent, derive a formula for the solution of minimal 2-norm.
(b) If the columns of \( A \) are linearly independent, derive a formula for the \( x \) that minimizes the 2-norm of the residual \( r = b - Ax \).
(c) Use the singular value decomposition of \( A \) to give a formula for the \( x \) that minimizes the 2-norm of the residual, and among all the \( x \)'s which minimize the 2-norm, it has minimal norm.

3. (a) Given a vector \( x \in \mathbb{R}^n \) and a natural number \( k < n \), give a formula for a unit vector \( w \) for which the vector \( y = (I - 2ww^T)x \) satisfies the following conditions: \( y_i = x_i \) for \( i < k \) and \( y_i = 0 \) for \( i > k \).
(b) Using these Householder transformations, write a pseudo code (or a Matlab code) for reducing a real symmetric matrix to tridiagonal form using Householder similarity transformations.

4. (a) State and prove the Gerschgorin Circle Theorem.
(b) Let

\[
A = \begin{pmatrix}
5 & 1 & 3 \\
2 & 4 & 1 \\
3 & -1 & 4
\end{pmatrix}
\]

What can you say about the location of the eigenvalues of \( A \)?
(c) Use the Gerschgorin Circle Theorem to estimate the size of the largest root of the polynomial \( p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0 \).
5. (a) Find the ellipse or hyperbola of the form \( ax^2 + by^2 = 1 \) that best fits \( n \) data points \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\) in the plane by computing the least solution to the \( n \) linear equations gotten by substituting each of the data points into the quadratic equation.

(b) Test your method with the dataset \((1,1), (2,4), (3,9), (4,16)\).

**Part II: Numerical Analysis**

6. (a) State Newton’s method (the algorithm) for solving \( f(x) = 0 \) where \( f: R \to R \).

(b) Assuming \( f \) is smooth and we have an initial guess sufficiently close to a root \( p \), state sufficient condition(s) for the method to converge quadratically.

(c) Show that Newton’s method applied to \( x^5 = 2 \) converges quadratically to the root \( 2^{1/5} \) from any starting guess \( x > 0 \).

7. (a) Given a function \( f \) with \( n + 1 \) continuous derivatives on the interval \( I = [-1,1] \), show that, given \( n + 1 \) points \( \{x_0, x_1, \ldots, x_n\} \) in \( I \), there exists a unique polynomial \( P_n(x) \) of degree \( \leq n \) such that

\[
P_n(x_i) = f(x_i) \quad \text{for} \quad i = 0 \ldots n.
\]

(b) Prove that, given \( t \in I \), there exists \( \eta \in I \) such that

\[
f(t) - P_n(t) = \frac{(t - x_0) \cdots (t - x_n) f^{(n+1)}(\eta)}{(n+1)!}.
\]

(c) From the above we get that (all norms are on the interval \( I \))

\[
\|f - P_n\|_\infty \leq \max_{t \in I} |(t - x_0)(t - x_1) \cdots (t - x_n)| \frac{\|f^{(n+1)}\|_{\infty}}{(n+1)!}.
\]

What choice for the points \( x_0, \ldots, x_n \) minimizes the right hand side of this error bound?

8. (a) Find a function \( q(x) \) which has a continuous first derivative for each \( x \in (-\infty, \infty) \) with the properties:

\[
q(x) = \begin{cases} 
1 & \text{if } x = 0 \\
0 & \text{for all } |x| \geq 2 \\
a_0 + a_1 x + a_2 x^2 & \text{for all } x \in [-2, -1] \\
b_0 + b_1 x + b_2 x^2 & \text{for all } x \in [-1, 1] \\
c_0 + c_1 x + c_2 x^2 & \text{for all } x \in [1, 2]
\end{cases}
\]

(b) Use the function \( q(x) \) defined in part a to interpolate a set of equally spaced points \((x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n), \) where \( h = x_{k+1} - x_k \) for all \( k \). In particular, describe how to find constants \( c_k \) so that the function \( Q(x) = \sum_{k=0}^{n} c_k q\left(\frac{x-x_k}{h}\right) \) has the property that \( Q(x_k) = y_k \) for all \( k \).

9. Let \( P_2(x) \) be a quadratic polynomial interpolating \( g(x) \) at \( x = 0, h, 2h \).

(a) Use this to derive a numerical integration formula \( I_h \) for

\[
I = \int_{-h}^{3h} g(x)dx.
\]
(b) Use a Taylor series expansion of \( f(x) \) to show

\[
I - I_h = \frac{3}{8} h^4 f^{(3)}(0) + O(h^5).
\]

10. **Triple Recursion Formula** If \( \{\phi_n(x)\} \) is an orthogonal family of polynomials on \([a, b]\), with respect to the weight function \( w(x) \geq 0 \), and \( n \geq 1 \), then show that \( \phi_{n+1}(x) = (a_n x + b_n)\phi_n(x) - c_n \phi_{n-1}(x) \), for some constants \( a_n, b_n, \) and \( c_n \). (Hint: Let \( g(x) = \phi_{n+1}(x) - a_n x \phi_n(x) \), where \( a_n \) is a constant chosen so that \( g(x) \) has degree \( n \).)