1. Consider the Householder matrix $H = I - 2ww^T$ where $w^Tw = 1$.
   (a) Show that $H$ is orthogonal.
   (b) Determine the eigenvalues of $H$.
   (c) Given a vector $x$ and an integer $k$, find a vector $w$ such that $(Hx)_i = x_i$ for $i < k$
       and $(Hx)_i = 0$ for $i > k$.
   (d) Explain why one needs to be careful about the choice of sign in the formula for $w$.

2. Let $A$ be a symmetric, positive definite matrix. Show that if $A$ is $LU$ factored,
then the elements of $U$ in absolute value are bounded by the largest diagonal element
of $A$.

3. Give a careful statement and proof of Gershgorin's Theorem. Find upper and
lower bounds for the set of eigenvalues for the matrix
\[
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2
\end{pmatrix}
\]

4. Let $A$ be a nonsingular $n \times n$ matrix and $\{X_k\}$ a sequence of $n \times n$ matrices
   generated by the iteration:
   \[X_{k+1} = X_k + X_k(I - AX_k).\]
   Show that if $X_0$ is chosen such that the spectral radius of $I - AX_0$ is strictly less
   than 1, then $\{X_k\}$ converges to the inverse of $A$.

5. For a given partition $\{x_0, x_1, \ldots, x_n\}$ of an interval $[a, b]$ and function $f$ defined
   on $[a, b]$
   (a) State the divided difference formula for the polynomial $p_n(x)$ interpolating $f$
       from the data $(x_k, f(x_k)), k = 0, 1, \ldots, n$.
   (b) Show that the error in the approximation in part (a) is given by
       \[f(x) - p_n(x) = (x - x_0)(x - x_1) \cdots (x - x_n)f[x_0, x_1, \ldots, x_n, x]\]
   (c) For $\{x_0, x_1, x_2\} = \{0, 1, 2\}$ and $f(x) = 1/(1+x^2)$, compute the divided difference
       formula for $p_2(x)$. 

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6. Let $A$ be a real, positive definite $n \times n$ matrix and $b$ be an $n \times 1$ vector.
(a) Derive the steepest descent algorithm for approximating the true solution $x$ of $Ax = b$, by a sequence of iterates $\{x_k\}$.
(b) Define the real-valued function $h$ on $\mathbb{R}^n$ by

$$h(x) = (Ax - b)^T A^{-1} (Ax - b)$$

and show that

$$h(x_{k+1}) \leq (1 - c(A)^{-1})^2 h(x_k)$$

where $c(A)$ is the condition number of the matrix $A$.

7. For an algorithm, define the terms inherent error, total effect of rounding, and numerically stable.

Carefully formulate a numerically stable algorithm for computing the positive solution of the quadratic equation $x^2 + 2px - q = 0$ where $p$ and $q$ are arbitrary positive numbers. Justify the numerical stability of your algorithm.

8. Give a careful proof that the DFT of a convolution of two finite sequences (not necessarily of the same length) is proportional to the product of the DFT's of the original sequences, suitably padded.

9. Given the $N$ sampled values $f(2k\pi/N), \ k = 0, 1, 2, \ldots, N - 1$, of a twice continuously differentiable, $2\pi$-periodic function $f$, outline a method of computing the Fourier coefficients of $f$ which has the following properties

- makes use of the FFT routine
- produces an estimate which improves as $N \to \infty$.

Estimate the accuracy of your method.

If possible, state an explicit formula for its implementation.