Directions: You have 2 hours to answer the following questions. You must show all your work as neatly and clearly as possible and indicate the final answer clearly. You may not use a calculator.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Possible</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>25</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>
(1) (25 points)
(a) Show that the equation \( x = \frac{1}{2} \cos(x) \) has a solution \( \alpha \).

(b) Find an interval \([a, b]\) containing \( \alpha \) and such that for every \( x_0 \in [a, b] \), the iterative sequence

\[
x_{n+1} = \frac{1}{2} \cos(x_n)
\]

will converge to \( \alpha \). Justify your answer.
(2) (15 points) For the basic Lagrange polynomials

\[ L_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} \quad \text{for} \quad i = 0, \ldots, n, \]

show that

\[ \sum_i L_i(x) = 1 \quad \text{for all} \quad x. \]
(3) (20 points) Consider a quadrature rule of the form for $0 < \alpha < 1$:

$$\int_0^1 x^\alpha f(x) \, dx \approx A \int_0^1 f(x) \, dx + B \int_0^1 xf(x) \, dx.$$ 

(a) Determine the constants $A, B$ so that the quadrature formula has maximum degree of exactness.

(b) What is the degree of exactness of the formula in part (a)?
(4) (20 points) Let $f$ be an arbitrary (continuous) function on $[0, 1]$ satisfying

$$f(x) + f(1 - x) \equiv 1 \quad \text{for} \quad 0 \leq x \leq 1.$$ 

(a) Show that $\int_0^1 f(x) \, dx = \frac{1}{2}$.

(b) Show that the composite trapezoidal rule for computing $\int_0^1 f(x) \, dx$ is exact.
(5) (20 points) Assume that you are solving the initial value problem.

\[ y' = f(t, y) \quad a \leq t \leq b \]
\[ y(a) = \alpha \]

The formula for the global error of the numerical solutions for the ODE problem above obtained via Euler’s method is \((M = ||Y''||_\infty)\):

\[ |Y(t_i) - w_i| < \frac{hM}{2L} [e^{L(b-a)} - 1]. \]

Compute the values of \(L\) and \(M = ||Y''||_\infty\) necessary to apply the global error formula above to the specific ODE problem

\[ y' = \sin(t + 2y) + e^t \quad 0 \leq t \leq 1 \]
\[ y(0) = 0 \]