PhD Complex Analysis Exam, January 3, 2014.

Do any 9 problems.

The notation \( f \in H(\Omega) \) means \( f \) is holomorphic in the domain \( \Omega \).

1. Prove the fundamental theorem of algebra.

2. Let \( \Omega \) be a domain on the complex plane and \( M \) be a finite positive constant. Suppose \( F \) is a family of holomorphic functions on \( \Omega \). If \( |f(z)| \leq M \) for all \( z \in \Omega \) and all \( f \in F \), prove that \( F \) is equi-continuous on \( \Omega \).

3. Let \( f_n \in H(\Omega), n = 1, 2, 3, \ldots \) be one to one in \( \Omega \). If \( f_n \to f \) uniformly on compact subsets in \( \Omega \), prove that \( f \) is either one to one or constant. Show with examples that both conclusions can occur.

4. Let \( f \in H(\Omega) \). If \( f \) has no zeros in \( \Omega \), prove that \( \ln |f| \) is harmonic in \( \Omega \).

5. Evaluate the integral
\[
\int_0^\infty \frac{\ln x}{(x^2 + b^2)^2}, b > 0.
\]

6. Let \( \mathbb{C} \) be the complex plane. Suppose \( f : \mathbb{C} \to \mathbb{C} \) is continuous and \( f \) is holomorphic except in \([-1, 1]\). Prove that \( f \) is entire.

7. Prove that a doubly periodic entire function is constant.

8. Let \( P \) be a polynomial and \( C \) the circle \(|z - a| = R\). Evaluate the integral \( \int_C P(z)dz \).

9. Let \( f, F \in H(U), \) where \( U = \{z : |z| < 1\} \) and let \( R_f = \{f(z) : z \in U\} \) and \( R_F = \{F(z) : z \in U\} \). Suppose \( F \) is one to one in \( U \), \( f(0) = F(0) \) and \( D_f \) is contained in \( D_F \). Prove that there exists an \( \omega \in H(U) \) such that \( f(z) = F(\omega(z)) \) and \( |\omega(z)| \leq |z| \). Also show that the equality holds if and only if \( D_f = D_F \).

10. Suppose \( f \in H(\Pi^+) \), where \( \Pi^+ \) is the upper half plane and \( |f| \leq 1 \). How large can \( |f'(i)| \) be? Find the extremal functions.

11. Find all entire functions \( f \) such that \( |f(z)| = 1 \) whenever \( |z| = 1 \).