PH.D. QUALIFYING EXAM IN COMPLEX ANALYSIS

Give complete proofs and computations. Partial credit will be given where justified. In the following, \( \mathbb{C} \) denotes the set of complex numbers and \( D = \{ z \in \mathbb{C} \mid |z| < 1 \} \).

1) Evaluate the integral
\[
\int_0^\infty \frac{\log x}{(1 + x^2)^2} \, dx.
\]

2) (a) Let \( f \) be analytic at \( z \in \mathbb{C} \). Prove that
\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 = 4|f'(z)|^2.
\]
(b) Let \( f_1, f_2, \ldots, f_n \) be analytic in the region \( G \). Prove that \( |f_1|^2 + |f_2|^2 + \ldots + |f_n|^2 \) is harmonic on \( G \) if and only if \( f_k \) is constant, for all \( k = 1, \ldots, n \).

3) Let \( \{ f_n \} \) be a sequence of functions, each analytic in the open set \( G \), which converges to \( f \) uniformly on all compact subsets of \( G \). Prove that \( f \) is analytic in \( G \).

4) Let \( P(z) = z^7 + z^5 + 5z^3 + 1 \). Find the number of zeros of \( P \) counted according to their multiplicities in
   (a) \( \{ z \in \mathbb{C} \mid |z| < 1 \} \)
   (b) \( \{ z \in \mathbb{C} \mid 1 < |z| < 2 \} \)
   (a) \( \{ z \in \mathbb{C} \mid |z| > 0 \} \)

5) Let \( G \) be a region and \( f \) be analytic in \( G \). Prove that if \( f(G) \) is a subset of a circle, then \( f \) is a constant function.

6) Suppose \( f \) is a nonconstant analytic function in \( D \) such that \( |f(z)| \leq 1 \). Let \( a = f(0) \). By considering the function
\[
g(z) = \frac{f(z) - a}{1 - \overline{a}f(z)},
\]
prove that
\[
\frac{|a| - |z|}{1 + |a||z|} \leq |f(z)| \leq \frac{|a| + |z|}{1 - |a||z|}
\]
for \( |z| < 1 \).

7) (a) Does there exist an analytic mapping \( f : D \to D \) such that \( f(0) = 0 \) and \( f\left(\frac{1}{2}\right) = \frac{1}{3} \)? Justify your answer.
   (b) Does there exist an analytic mapping \( f \) of \( \{ z \in \mathbb{C} \mid \text{Re}(z) > 0 \} \) into itself such that \( f(3) = 3 \) and \( f(9) = 6 \)? Justify your answer.

8) Let \( \mathcal{F} \) be the collection of all analytic mappings of \( D \) into \( \{ z \in \mathbb{C} \mid \text{Re}(z) > 0 \} \) such that \( f(0) = 1 \). Show that \( \mathcal{F} \) is a normal family.