1. Let $a_n$ denote the number of strings of the letters A, B, C, and D such that the letter A appears an odd number of times.

   (a) Find a closed formula for $a_n$.

   (b) Find the exponential generating function for the sequence $\{a_n\}$.
   (using no summation signs)
2. The set \([n] \times [n]\) is partially ordered by the relation \((a, b) \preceq (c, d)\) which holds when \(a \leq c\) and \(b \leq d\). Find, with proof, the length of a maximum chain and the length of a maximum antichain.

What is the size of a minimum chain decomposition of this poset. (Recall that a chain decomposition of a poset is a partition of its elements into disjoint chains.)
3. (a) Prove that, for a simple graph $G$ with at least 5 vertices, at least one of $G$ or its complement has a cycle.

(b) Suppose that you color the edges of $K_n$ using 2 colors. Show that there exists a monochromatic spanning tree.
4. A *parity check matrix* $H$ of a binary linear code $C$ can be defined as a generator matrix of the dual code $C^\perp$.

Show that $C$ is the null space of the transpose $H^T$, where multiplication by $H^T$ is on the right, i.e., $cH^T$.

Show that the minimum distance of the code equals the cardinality of a minimum dependent set of columns of $H$.

If a generator matrix of a binary linear code $C$ is

$$
\begin{pmatrix}
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 0
\end{pmatrix},
$$

then show that $(c_1, c_2, c_3, c_4, c_5)$ is a codeword if and only if (modulo 2)

$$
c_1 + c_2 + c_3 = 0 \\
c_2 + c_4 = 0 \\
c_1 + c_5 = 0.
$$

What is the minimum distance of this code?
5. Let \( g(n) \) be the number of permutations of length \( n \) in which each cycle is of even length. Find the exponential generating function of the sequence \( g(n) \), where \( n = 0, 1, 2, \cdots \).
6. Let \( t_n \) be the total number of cycles of all permutations of length \( n \). So \( t_1 = 1 \), \( t_2 = 3 \), and \( t_3 = 11 \). Find an explicit formula for the numbers \( t_n \). Your answer can contain one summation sign.
7. Prove that the language \( \{a^{n^2} : n \in \mathbb{N}\} \) is not regular.
8. Let $\beta$ be a permutation of length $k$. Prove that there are precisely $k^2 + 1$ permutations of length $k + 1$ containing $\beta$. 