Combinatorics Exam May 2011

1. Let $f(n)$ be the number of permutations of length $n$ that have no cycle that is shorter than three.
   
   (a) Find the exponential generating function of the sequence $f(n)$. You can assume that $f(0) = 1$.
   
   (b) Find $\lim_{n \to \infty} f(n)/n!$.

2. We divide a group of $n$ people into subgroups $A$, $B$, and $C$, and ask each subgroup to form a line. We also require that $A$ have an odd number of people, and that $B$ have an even number of people. How many ways are there to do this?

3. Let $A$ be the graph obtained from $K_n$ by deleting an edge. Find a formula for the number of spanning trees of $A$.

4. Let $X_n(p)$ be the number of cycles of the $n$-permutation $p$. Compute $VAR(p)$.

5. Show that a planar graph for which every face has an even number of edges must be bipartite.

6. Recall that a tournament is a complete directed graph. Prove that there exists a tournament on eight vertices that contains at least 316 Hamiltonian paths.

7. For which values of $m$ and $n$ is the complete bipartite graph $K_{m,n}$ planar, for which values of $m$ and $n$ is it Hamiltonian, and for which values of $m$ and $n$ is it Eulerian?

8. Prove in a finite poset $P$, the number of elements in the longest chain is equal to the smallest number $k$ so that $P$ can be decomposed into the union of $k$ antichains.