Show your work.

1a. There are \( r \) black and \( n - r \) white balls in an urn. They are removed one at a time without replacement. What is the probability that exactly \( k \) drawings are required to get a white ball?

b. Use your result to conclude that

\[
\sum_{k=1}^{r} \frac{(r)_k}{(n-1)_k} = \frac{r!}{n! (n-r)!} = \frac{n}{n-r}
\]

2. Each of \( n \) people is to be mailed an envelope containing a letter and an bill. How many ways \( Q_n \) are there of placing the \( n \) letters and \( n \) bills into \( n \) addressed envelopes so that no envelope contains both the correct letter and bill?

3. Let \( P_n \) be the total number of \( k \)-permutations of \( n \) for various \( k \), that is,

\[
P_n = \sum_{k=0}^{n} (n)_k, \quad n = 0, 1, \ldots
\]

Show that

\[
P(t) = \sum_{n=0}^{\infty} P_n \frac{t^n}{n!} = (1 - t)^{-1} e^t
\]

and use this to show that

\[
P_n = nP_{n-1} + 1, \quad n = 1, 2, \ldots, P_0 = 1.
\]

4. Define the binary Hamming code \( H(r) \) of length \( 2^r - 1 \). Show that \( H(r) \) is an exactly single error correcting code and that \( H(r) \) is a perfect code. Determine the number of codewords of weight 3 in \( H(r) \).

5. Show that the number of partitions of a number \( n \) into exactly \( m \) parts is equal to the number of partitions of \( n - m \) into no more than \( m \) parts.

6. An order on the set of ordered pairs of non-negative integers is defined by \((a_1, a_2) \leq (b_1, b_2)\) if \( a_i \leq b_i \) for \( i = 1, 2 \). Find the Möbius function of this poset.

7. Determine for which values of \( m \) and \( n \) the complete bipartite graph \( K_{mn} \) is (a) planar, (b) Eulerian, (c) Hamiltonian.

Answer the same three questions for the \( n \)-cube. (Recall that the \( n \)-cube \( Q_n \) is defined as the graph whose vertices are the set of all binary sequences of length \( n \).)
where two vertices are adjacent if the corresponding sequences differ by exactly one digit.

8. Let $G$ be a triangle-free graph with $n$ vertices, minimum degree $k$ and girth $g$. Prove that $g \leq 2n/k$.

Hint: For an appropriate cycle $C$ count the number of edges from $C$ to $G \setminus C$ in two ways.