COMBINATORICS PhD EXAMINATION
May 24, 1995

1. (a) Let $A_1, A_2, \ldots, A_k$ be subsets of $[n]$ such that $A_i \cap A_j \neq \emptyset$, for all $i, j$. Show that

$$k \leq 2^{n-1}.$$ 

Is this inequality best possible?

(b) How many pairs $(A, B)$ of subsets of $[n]$ are there such that $A \cap B = \emptyset$?

2. Use inclusion–exclusion to determine the number of monic polynomials of degree $n$ with no roots in $\mathbb{Z}_p[x]$.

3. (a) Prove that the number of partitions of $n$ into at most $k$ parts equals the number of partitions of $n$ into parts of size of most $k$.

(b) Prove that the number of partitions of $n$ into odd parts is equal to the number of partitions of $n$ into unequal parts. (Hint: use generating functions).

4. Prove that the number of $k$–dimensional subspaces of the $n$ dimensional vector space over $\mathbb{F}_q$ is

$$\binom{n}{k}_q = \frac{(q^n - 1) \ldots (q^{n-k+1} - 1)}{(q^k - 1) \ldots (q - 1)}$$

5. Prove that $K_{33}$ is not planar.

6. Prove that every regular bipartite graph has a perfect matching. Show that this is not true in general for regular graphs of even order.

7. Prove that in a binary self dual code, either all the vectors have weight divisible by 4 or half have even weight not divisible by 4 and half have weight divisible by 4.

8. Let $H$ be a parity check matrix for a binary $(n, k, d)$– code $C$ with $n \geq 4$. If the columns of $H$ are distinct and all have odd weight, then $d \geq 4$.

9. (a) Show the nonexistence of a difference set with parameters $(31, 10, 3)$.

(b) Show the nonexistence of a symmetric design with parameters $(29, 8, 2)$. (Hint: consider the appropriate equation modulo 3)

(c) Show the nonexistence of a Steiner system $S(3, 6, 11)$.

(d) Find a $(13, 4, 1)$ difference set $D$. The set of translates of $D$ form the blocks of a design. Find the parameters $b, v, k, r, \lambda$.

10. Let $A$ be an affine plane of order $n$. Write $B_1 \sim B_2$ if lines $B_1$ and $B_2$ are the same or have no points in common. Show that $\sim$ is an equivalence relation. Show that there exists a projective plane $\pi$ such that $A$ is obtained from $\pi$ by deleting one line and all the points on that line.