Do 4 out of 5 problems.

1. A \( k \)-arc of a projective plane \( \Pi \) is a set of \( k \) points, no three of which are collinear. Prove that every 4-arc in the plane \( PG(2, 4) \) lies in exactly two 5-arcs.

2. Use (without proof) the assertion of problem #1 to prove that the 5-arcs of \( \Pi = PG(2, 4) \) form a block design \( \Sigma \) on the point set of \( \Pi \); with \( v = 21 \) and \( k = 6 \). Compute \( \lambda \).

3. State the Hall Multiplier Theorem, and use it to find a \((91, 10, 1)\) difference set.

4. HC is the problem of deciding whether a graph has a Hamiltonian circuit or not, and this problem is known to be NP-complete. Use this fact to prove that HP is also NP-complete, where HP is the problem of deciding whether a graph has a Hamiltonian path (with unspecified initial and terminal vertices).

5. Let \( E \) be a finite set and \( P \) a partition of \( E \). Call a subset \( I \) of \( E \) independent \( (I \in \mathcal{I}) \) if no two elements of \( I \) are in the same block of \( P \).

1. Prove that \((E, \mathcal{I})\) is a matroid (called a partition matroid).
2. For a bipartite graph \( B \) with edge set \( E \) let \( \mathcal{M} \subseteq 2^E \) denote the set of matchings on \( B \). Show that \((E, \mathcal{M})\) is not a matroid.
3. Prove that \( \mathcal{M} \) is the intersection of two partition matroids. In other words there are partition matroids \((E, \mathcal{I}_1)\) and \((E, \mathcal{I}_2)\) such that \( \mathcal{M} = \mathcal{I}_1 \cap \mathcal{I}_2 \).

Part 3 is significant because there is a general result that states that if \((E, \mathcal{M})\) is such a matroid intersection then there is a polynomial algorithm to find a maximum cardinality independent set.