Ph.D. Exam in Analysis   May 2012

Be sure to carefully present all work in a neat and logical fashion. Do not leave any gaps. State clearly theorems used in your proofs. Print your name on each sheet turned in.

1. Let $\mu$ be a real-valued signed measure on a $\sigma$-algebra $\Sigma$. Does there exist a set in $\Sigma$ on which $\mu$ attains its minimum value? Prove or disprove.

2. Let $X$ be a Banach space. Show that there exists a compact $T_2$ space $S$ such that $X$ is isometrically isomorphic to a closed linear subspace of $C(S)$. [Hint: Examine the unit ball of the dual space $X^*$ with the appropriate topology and the action of $X$ on $X^*$]

3. Let $\{f_n\}$ be a sequence of integrable functions on $S$ such that $\sum \int |f_n| d\mu < \infty$. Give all details concerning the convergence of $\sum f_n(s), s \in S$.

4. State and prove the Fubini theorem.

5. Let $\pi$ be the natural map from the Banach space $X$ into its second dual $X^{**}$. Show $\pi(X)$ is closed in the norm topology of $X^{**}$.

6. Show that the continuous functions on $[0, 1]$ are dense in $L^1[0, 1]$ (Lebesgue measure).
   [Hint: First show this is true for the indicator function of a Lebesgue measurable subset of $[0, 1]$]