Analysis
PhD Examination
January 2012

Answer SIX questions. Write solutions in a neat and logical fashion, giving complete reasons for all steps and stating carefully any substantial theorems used.

1. Let $V$ be the vector space of all complex sequences $z = (z_n)_{n \in \mathbb{N}}$ and for each $n \in \mathbb{N}$ denote the $n$th coordinate map by

$$\phi_n : V \to \mathbb{C} : z \mapsto z_n.$$

Does $V$ carry a norm relative to which the linear functional $\phi_n$ is bounded for infinitely many values of $n$? Explain.

2. State the Baire Category Theorem.
   Decide whether the vector space $c_0$ comprising all finitely-nonzero complex sequences carries a complete norm (by considering suitable subspaces of finite dimension, or otherwise).

3. State the Closed Graph Theorem.
   Prove the Banach Isomorphism Theorem.

4. Let $1 \leq p < \infty$ and define norms $\| \cdot \|_p$ and $\| \cdot \|_\infty$ on $C[0, 1]$ by

$$\|f\|_p = \left( \int_0^1 |f(t)|^p dt \right)^{1/p}$$

$$\|f\|_\infty = \sup\{|f(t)| : 0 \leq t \leq 1\}$$

as usual. Show that on $C[0, 1]$:
   (i) $\| \cdot \|_p$ and $\| \cdot \|_\infty$ are inequivalent;
   (ii) $\| \cdot \|_p$ is incomplete.

5. Let $\mathbb{H}$ be a complex Hilbert space and $\phi : \mathbb{H} \to \mathbb{C}$ a bounded linear functional. Prove that there exists a unique $u \in \mathbb{H}$ such that $\phi(v) = \langle u | v \rangle$ for each $v \in \mathbb{H}$.
6. Say what it means for one measure to be absolutely continuous relative to another.

Let \((\mu_n)_{n=1}^\infty\) be a sequence of finite measures on the \(\sigma\)-algebra \(\mathcal{F}\). Prove that there exists a finite measure \(\lambda\) on \(\mathcal{F}\) such that \(\mu_n \ll \lambda\) for each \(n \geq 1\).

7. Let \((\Omega, \mathcal{F}, \mu)\) be a finite measure space on which \(f\) is a measurable real-valued function. Prove that the rule

\[
    t \in \mathbb{R} \Rightarrow F(t) = \int_{\Omega} e^{itf(\omega)} d\mu(\omega)
\]

defines a continuous function \(F : \mathbb{R} \to \mathbb{C}\).

8. Let \((\Omega, \mathcal{F}, \mu)\) be a \(\sigma\)-finite measure space, let \(g : \Omega \to \mathbb{C}\) be measurable and let \(p \geq 1\). Show that pointwise multiplication

\[
    f \in L_p \Rightarrow \Phi_g(f) = gf
\]

defines a bounded linear operator \(\Phi_g\) on \(L_p\) iff \(g \in L_\infty\). Note: Recall that \(g \in L_\infty\) iff there exists \(K \geq 0\) such that \(\mu\{|g| > K\} = 0\).