Functional Analysis Qualifying Exam, May 2003

Do All Nine

1. Suppose $X, Y$ are topological vector spaces, $\Lambda : X \to Y$ is linear, and $N$ is a closed subspace of $X$. Let $\pi : X \to X/N$ denote the quotient map. Show, if $\Lambda(x) = 0$ for each $x \in N$, then there is a unique $f : X/N \to Y$ satisfying $\Lambda = f \circ \pi$. Show further, $f$ is linear and $\Lambda$ is continuous if and only if $f$ is continuous.

2. Show, if $X$ is an infinite dimensional $F$-space (a topological vector whose topology is induced by a complete invariant metric), then $X$ does not have a countable Hamel basis. A Hamel basis is a basis in the sense of linear algebra.

3. Is every weakly convergent sequence in $\ell^1$ stongly convergent? Are the weak and strong topologies on $\ell^1$ the same?

4. Let $\delta$ denote the distribution defined by $\delta(\phi) = \phi(0)$ for $\phi \in D(\mathbb{R})$. Find the support of $\delta'$. For which $f \in C^\infty(\mathbb{R})$ is $f\delta' = 0$. Is it possible that an $f \in C^\infty(\mathbb{R})$ can vanish on the support of an $\Lambda \in D'(\mathbb{R})$ and yet $f\Lambda \neq 0$?

5. Construct a sequence in $D(\mathbb{R})$ which converges to 0 in the topology of $S_1$ (tempered distributions), but not in the topology of $D(\mathbb{R})$.

6. Prove, if $\mathcal{A}$ is a finite dimensional Banach algebra with unit $e$, if $x, y \in \mathcal{A}$, and if $xy = e$, then $yx = e$.

7. Show, if $X$ is a compact Hausdorff space, then there is a natural one-one correspondence between closed subsets of $X$ and ideals in $C(X)$.

8. Let $T = \{\gamma \in C : |\gamma| = 1\}$ denote the unit circle. Let $U$ denote the bilateral shift, viewed as the operator $Uf(\gamma) = \gamma f(\gamma)$ on $L^2(T)$ (with respect to arclength measure). Given $\omega$ a measurable subset of $T$, let $P_\omega$ denote the projection onto the subspace of functions which are supported in $\omega$. Does $P_\omega$ commute with $U$? Find the spectral decomposition of $U$.

9. Let $\mathcal{A}$ denote the commutative Banach algebra of functions of the form $f(x) = \sum_{n \in \mathbb{Z}} f_n \exp(inx)$, with $\sum |f_n| < \infty$ under pointwise multiplication and with the norm $\|f\| = \sum |f_n|$. Show, if, for each $x$, $f(x) \neq 0$, then $f$ is invertible in $\mathcal{A}$. Is $\mathcal{A}$ a $C^*$-algebra under pointwise complex conjugation?