PhD EXAM IN ANALYSIS. May 2002.

I) State the following theorems:

1. The Egorov theorem
2. The Radon-Nikodym theorem
3. Density of step functions in $L^1$ and $L^\infty$
4. The Fatou theorem for $\mu$-integrable functions
5. The Riesz representation theorem of the dual of $C(K)$.

II) State and prove one of the following theorems:

1. The integral representation of the dual of $L^1$ (the finite measure case).
2. The theorem of integration with respect to $\mu$.

III) Solve the following problems:

1. Let $(X, \Sigma, \mu)$ be a measure space with $\mu(X) < \infty$, a Banach space and $\mathcal{F} \subseteq \Sigma$ a $\sigma$-algebra. Prove the existence of the conditional expectation $E(f|\mathcal{F})$ for $f \in L^1(\mu)$.
2. Let $(X, \mathcal{F}), (Y, \mathcal{T})$ be measurable spaces and $\mu: \mathcal{F} \to \mathbb{R}$, $\nu: \mathcal{T} \to \mathbb{R}$ $\sigma$-additive measures. Prove the existence of the product measure $\mu \times \nu$ and the equality $1_{\mathcal{F} \times \mathcal{T}} = 1_{\mathcal{F}} \times 1_{\mathcal{T}}$.
3. Let $(X, \mathcal{F}), (Y, \mathcal{T})$ be measurable spaces, $p: X \to Y$ a $(\mathcal{F}, \mathcal{T})$ measurable mapping, $\mu: \mathcal{F} \to \mathbb{R}_+$ a $\sigma$-additive measure and $\nu = p_*(\mu)$, the image measure of $\mu$ under the mapping $p$. Prove that if $f \in L^1(\nu)$, then $f \circ p \in L^1(\mu)$ and $\int_Y f \circ p \, d\mu = \int_X f \, d\mu(p(x))$. 
4. Let \((X, \Sigma, \mu)\) be a measure space and \(f \in L^1(\mu)\). Prove that: For every \(\varepsilon > 0\), there is a set \(X_\varepsilon \in \Sigma\) with \(\mu(X_\varepsilon)\), such that
\[
\int_{X \setminus X_\varepsilon} |f| \, d\mu < \varepsilon.
\]

5. Let \(\lambda\) be the Lebesgue measure on \([0, 1]\) and \(f \in L^1(\lambda)\). Prove that if
\[
\int_{[0,1]} x^n f \, d\lambda = 0 \quad \text{for every } n,
\]
then \(f = 0, \lambda\text{-a.e.}\).

Hint: Use the fact that the continuous functions on \([0, 1]\) are dense in \(L^1(\lambda)\) and reduce to the case \(\int_A f \, d\mu = 0\) for \(A \in \mathcal{B}([0, 1])\).

Note: Write complete computations and explanations. Do not use arrows or other unnecessary signs. Use words for explanations. Write complete sentences. State the complete framework for each theorem.