I. State the following theorems:

1. The monotone convergence theorem
2. The Hahn decomposition theorem
3. The Egorov theorem
4. The Vitali convergence theorem
5. The Radon-Nikodym theorem
6. The Fubini theorem
7. The representation of the dual of $L^p$
8. The Lebesgue dominated convergence theorem

II. Prove one of the theorems 6 or 7.

III. Solve 5 of the following problems

1) In what follows, $(X, \mathcal{E}, \mu)$ is a measure space.

1) Let $(f_n)$ be a sequence of real valued measurable functions on $X$ and $f$ a real measurable function on $X$. Prove that $(f_n)$ converges to $f$ in $\mu$-measure if every subsequence $(f_{n_k})$ contains a further subsequence $(f_{n_{k_l}})$ converging to $f$, $\mu$-a.e.

2) Let $(f_n)$ be a sequence of positive, $\mu$-integrable functions such that $\sum_n \int_X f_n \, d\mu < \infty$. For each $x \in X$ denote

$$L(x) = \sup \{ n ; f_n(x) > 0 \}$$

Prove that $L(x) < \infty$, $\mu$-a.e.

3) Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Prove that for every $\varepsilon > 0$, there is a bounded interval $I$ such that $\int_E |f| \, dx / |I| < \varepsilon$ for every measurable set $E$ with $E \cap I = \emptyset$. 
4.) Let $F$ be a Banach space and $\mathcal{F}$ be a sub-$\sigma$-algebra. Prove the existence of the conditional expectation $E(f\mid\mathcal{F})$ for every $\mu$-integrable function $f: X \to F$.

5.) Let $(X, \mathcal{S}, \mu)$, $(Y, \mathcal{T}, \nu)$ be two real measure spaces. Prove the existence of the product measure $\mu \times \nu$ and that $1_{X \times Y} = 1_{\mu} \times 1_{\nu}$.

6.) Let $\mu$ be the Lebesgue measure on $\mathbb{R}$ and $f$ a Lebesgue integrable function on $\mathbb{R}$ such that $\int_{0}^{x} f \, d\mu = 0$ for every $x \in \mathbb{R}$ (if $x < 0$, $\int_{x}^{0} f \, d\mu = -\int_{0}^{x} f \, d\mu$). What can you say about $f$?

7.) Let $\mu$ be the Lebesgue measure on $\mathbb{R}$ and $f_n = -4^n \chi_{(-n, n)}$, $n = 1, 2, \ldots$. Show that $(f_n)$ is increasing but that the conclusion of the monotone convergence theorem is not true. Explain why.

8.) Let $\mu$ be the Lebesgue measure on $(0, 1)$. Each point $x \in (0, 1)$ has a binary expansion $x = \sum_{i=1}^{\infty} \frac{x_i}{2^i}$, where $x_i = 0$ or 1. (The expansion is unique except a countable set, where we can throw out of the space).

(i.) Define $\mathcal{F}_k(x) = x_k$. Sketch the graphs of $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$.

(ii.) Let $\frac{\mathcal{F}_k}{\mathcal{F}_k} = \sigma(\mathcal{F}_k, \mathcal{F}_2, \ldots, \mathcal{F}_{k-1})$. Describe $\mathcal{F}_k$.

(iii.) Let $f \in L^2(\mu)$. Find a function $g_1$ which is $\mathcal{F}_1$-measurable and satisfies $\int_X g_1 \, d\mu = \int_X f \, d\mu$ for $A \in \mathcal{F}_1$.

(i.e. Find $g_1 = E(f \mid \mathcal{F}_1)$. Find $g_2 = E(f \mid \mathcal{F}_2)$.)