Remark. In the sequel the word measure means positive, countably additive measure, unless otherwise stated.

Instructions. Do all of the following problems.

Problem 1. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of Lebesgue-measurable functions which converges pointwise to a function $g$. Prove, directly from the definition of measurability, that $g$ is measurable.

Problem 2. For $f \in L^1(\mathbb{R})$, let $f_t$ denote the translation of $f$ by $t$, i.e. $f_t(x) := f(x - t)$. Show that the map $t \mapsto f_t$ is a continuous map from $\mathbb{R}$ to $L^1(\mathbb{R})$.

Problem 3. Let $\mu$ denote the Lebesgue measure on $\mathbb{R}$. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = x^3 - x$. Define a Borel measure $\nu$ setting $\nu(A) := \mu(T^{-1}A)$. Compute the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$.

Problem 4. Let $K_n \subset [0, 1]$ be a Cantor set of Lebesgue measure larger than $1 - \frac{1}{n}$. Let $V := \bigcup_{n=1}^{\infty} K_n$. Prove that, for any nullset $N$, the set $V \setminus N$ is not a $G_\delta$ set. [Hint: Baire Category Theorem]

Problem 5. Let $\mu$ and $\nu$ be $\sigma$-finite measures on the measurable space $(X, B)$. Prove, from first principles, that $\nu$ can be written as $\nu = \nu_{ac} + \nu_{siag}$, with $\nu_{ac} \ll \mu$ and $\nu_{siag} \perp \mu$.

Problem 6. Let $x_n$ be a sequence in a Banach space $B$. Assume that that the sequence $x_n$ converges weakly to $x$. (a) Show that $x$ is the only weak limit of the sequence $x_n$. (b) Show that $\sup_n \|x_n\| < \infty$. [Hint for (b): Consider the $x_n$ as elements of the double dual $X^{**}$]

Problem 7. Let $K : [0, 2\pi] \to \mathbb{R}$ be defined by $K(x) = \frac{1}{4}\sin(x)$. Show that for every $h \in L^1([0, 2\pi])$, there exists a unique solution $f$ to the equation

$$f + f * K = h.$$

If $h$ is $C^\infty$, is it true that the solution $f$ is $C^\infty$? Justify your answer.

Problem 8. A distribution $T$ is called harmonic if $\Delta T = 0$, where $\Delta = \sum_1^n \frac{\partial^2}{\partial x_i^2}$. Show that if $T$ is a harmonic tempered distribution, then $T$ is a polynomial. [Hint: a distribution with support $\{0\}$ is a finite sum of multiples of Dirac's $\delta$ and its derivatives.]