1. Suppose $p_1, p_2, \ldots, p_r$ are distinct primes, and $K = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_r})$, find the Galois group of $K/\mathbb{Q}$. How many fields $F$ are there such that $\mathbb{Q} \subseteq F \subseteq K$?

2. (i) Show that if $K/k$ and $L/K$ are separable algebraic extensions, then so is $L/k$. (ii) Give an example of normal extensions $K/k$, $L/K$ such that $L/k$ is not normal.

3. Suppose $K$ is an algebraic extension field of $k$ and $\overline{k}$ is an algebraic closure of $k$. Show that $K/k$ is separable if and only if the ring $\overline{k} \otimes_k K$ has no nonzero nilpotent elements.

4. Suppose $A$ is a commutative ring with identity and $S \subset A$ is a multiplicative set not containing 0. (i) Show that there is a prime ideal $p \subset A$ such that $p \cap S = \emptyset$. (ii) Use part (i) to show that the intersection of all prime ideals of $A$ is the set of nilpotent elements.

5. Suppose $k$ is a field and $A$ is a finitely generated commutative $k$-algebra. Let $\overline{k}$ be an algebraic closure of $k$. Note that the Galois group $G = \text{Gal}(\overline{k}/k)$ acts on the set $\text{Hom}_k(A, \overline{k})$ of $k$-algebra homomorphisms $A \rightarrow \overline{k}$ by composition. Construct a bijection of the set of orbits of $G$ on $\text{Hom}_k(A, \overline{k})$ with the set of maximal ideals in $A$.

6. Suppose $A$ is an integral domain. Show that an ideal $I \subseteq A$ is invertible if and only if it is projective and finitely generated.

7. Suppose $X$ is a nonempty set and $F(X)$ is the free group on $X$. Show that $F(X)$ is abelian if and only if $X$ is a singleton.

8. (i) Define what it means for a group to be nilpotent. (ii) Show that if $G$ is a finite nilpotent group, then any Sylow subgroup is normal.

9. Suppose $A$ is a commutative ring with identity and $B$ is a commutative $A$-algebra with structure map $f : A \rightarrow B$. Show that the forgetful functor from the category of $B$-modules to the category of $A$-modules induced by $f$ has a left adjoint.

10. Let $p$ be a prime. Find all the isomorphism classes of noncommutative semisimple rings $A$ of cardinality $p^{10}$.

11. Suppose $R$ is a finite commutative ring with identity, such that $x^3 = x$ for all $x \in R$. Show that $R$ is a finite direct sum of fields isomorphic to $\mathbb{F}_2$ or $\mathbb{F}_3$. 