Name: ________________________________
Problems to be graded: 1 2 3 4 5 6 7 8 9 10 11

Note. Below ring means associative ring with identity, and module means unital module.

1. (10 points) Let \( f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x] \) be a polynomial with integer coefficients of degree \( n > 1 \). Suppose that for some \( k \), with \( 0 < k < n \) and some prime \( p \), we have \( p \nmid a_n; p \nmid a_k; p \mid a_i \) for \( i = 0, \ldots, k - 1 \); and \( p^2 \nmid a_0 \). Show that \( f(x) \) has a factor of degree at least \( k \) which is irreducible in \( \mathbb{Z}[x] \).

2. (10 points) Calculate the Galois group of \( x^5 - 12x + 2 \) over \( \mathbb{Q} \), the field of rational numbers. Justify your answer carefully.

3. (a) (5 points) Give an example of fields \( M \supseteq L \supseteq K \) such that \( M/L \) and \( L/K \) are normal extensions, but \( M/K \) is not normal. Justify your answer.
   (b) (5 points) Let \( L \supseteq K \) be fields such that \( [L : K] = 8 \). Prove that there exist \( \alpha, \beta, \gamma \in L \) such that \( L = K(\alpha, \beta, \gamma) \).

4. (10 points) Prove that the group defined by generators \( a \) and \( b \) and relations \( a^2 = b^3 = 1, \ abab = 1 \) is isomorphic to \( S_3 \).

5. (10 points) Let \( R \) be a ring with 1 and let

\[
0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0
\]

be a short exact sequence of (left) \( R \)-modules. Show that for any (right) \( R \)-module \( M \), the sequence

\[
M \otimes_R A \xrightarrow{1_M \otimes f} M \otimes_R B \xrightarrow{1_M \otimes g} M \otimes_R C \to 0
\]

of tensor products, is exact.

6. (10 points) State and prove Hilbert’s Basis Theorem.
7. (10 points) Prove the Lying-Over Theorem: Let $S$ be an integral extension of an integral domain $R$, and let $P$ be a prime ideal of $R$. Then, there exists a prime ideal $Q$ of $S$, such that $Q \cap R = P$.

8. (10 points) Prove that a commutative ring $R$ with identity is local if and only if, for all $r, s \in R$, $r + s = 1_R$ implies that either $r$ or $s$ is a unit.

9. Let $R$ be a ring.
   (a) (3 points) Define what it means for an $R$-module $M$ to be projective.
   (b) (4 points) Prove that any free $R$-module is projective.
   (c) (3 points) Prove that there exists some ring $R$ and some projective $R$-module $P$ such that $P$ is a projective module but not a free module.

10. (10 points) Determine up to isomorphism all semisimple noncommutative rings with $512 = 2^9$ elements.

11. Let $C$ be the category of all finitely generated $\mathbb{Z}$-modules, where $\mathbb{Z}$ is the ring of integers.
    (a) (3 points) Recall the definition of (direct) product in an arbitrary category $D$.
    (b) (4 points) Prove from your definition that, given a finite set $S$ of objects in $C$, there is a product $P$ of $S$ in $C$.
    (c) (3 points) Prove from your definition that there exists some set $T$ of objects in $C$, such that there is no product of $T$ in $C$. 