Ph.D. Algebra Exam – September 2003

Time allowed: 240 minutes

Do seven of the following eleven problems. Do not turn in more than seven problems. You must show your work. Answers with no work and/or no explanations will receive no credit. State clearly any theorem you use in your proofs.

In the problems, Z, resp. Q, C, is the set of all integers, resp. of all rational numbers, of all complex numbers.

1. Consider the following statement \( A(P) \): “If a normal subgroup \( H \) of a group \( G \) and the quotient \( G/H \) both have property \( P \), then so does \( G \).” Prove or disprove \( A(P) \), where
   a) \( P \) is “being solvable”;
   b) \( P \) is “being nilpotent”.

2. Inside the symmetric group \( S_7 \), consider the subgroup \( G \) generated by the permutations \((1234567)\) and \((235)(476)\). Let
   \[
   H = \langle a, b \mid a^7 = b^3 = 1, bab^{-1} = a^2 \rangle.
   \]
   Show that \( G \cong H \).

3. Let \( G \) be a finite group with exactly one maximal subgroup. Prove that \( G \) is cyclic of prime power order.

4. Let \( F \) be a finite field of finite cardinality \( q \) and \( n \) be any natural number. Show that there is at least one irreducible polynomial \( f \in F[x] \) of degree \( n \).

5. Let \( A \) be a commutative ring with identity. Suppose that \( M \) and \( N \) are free \( A \)-modules with \( m \) and \( n \) generators, respectively. Prove that if \( M \cong N \) then \( m = n \).

6. a) Give an example of a projective \( \mathbb{Z} \)-module and an example of an injective \( \mathbb{Z} \)-module.
   b) Let \( A \) be a finite abelian group considered as a \( \mathbb{Z} \)-module and let \( I \) be an injective \( \mathbb{Z} \)-module. Compute \( A \otimes \mathbb{Z} I \).

7. Let \( \alpha \in \mathbb{C} \) be such that \( [\mathbb{Q}(\alpha) : \mathbb{Q}] = 2003 \). Let \( E/\mathbb{Q} \) be a normal closure of \( \mathbb{Q}(\alpha)/\mathbb{Q} \). Prove that \( [E : \mathbb{Q}] \) divides 2003!

8. Let \( \xi \in \mathbb{C} \) be a primitive 77th root of unity in \( \mathbb{C} \), and let \( F = \mathbb{Q}(\xi) \).
   a) Briefly explain why \( F/\mathbb{Q} \) is a Galois extension, and describe the structure of the Galois group \( \text{Aut}_\mathbb{Q}(F) \).
   b) Find the number of subfields of \( F \) that are quadratic extensions of \( \mathbb{Q} \).

9. State and prove the Hilbert Basis Theorem.

10. Let \( D \) be a Dedekind domain and \( F \) be the group of fractional ideals of \( D \). Prove that any nontrivial element in \( F \) has infinite order.

11. Let \( R \) be a commutative ring with identity such that every element \( x \in R \) satisfies \( x^{n(x)} = x \) for some integer \( n(x) \geq 1 \) depending on \( x \). Prove that the Jacobson radical of \( R \) is 0.