Do seven of the following eleven problems. Please do not turn in more than seven problems.

You must show your work. Answers with no work and/or no explanations will receive no credit. State clearly any theorem you use in your proofs.

In the problems, \( \mathbb{Z} \), resp. \( \mathbb{Q} \), \( \mathbb{C} \), is the set of all integers, resp. of all rational numbers, of all complex numbers.

1. Let \( G \) be a finite simple (abelian or non-abelian) group of order \( n \). Find the number of normal subgroups of \( G \times G \).

2. Show that there is no simple group of order 112. (Caution: You are not allowed to use Burnside's \( p^a q^b \)-Theorem.)

3. Let \( N \) be a minimal normal subgroup of a finite group \( G \). Assume \( N \) is solvable. Show that \( N \) is an elementary abelian \( p \)-group for some prime \( p \).

4. Let \( A, B \) be two matrices over a field \( F \). Suppose that \[
\begin{pmatrix}
A & 0 \\
0 & A
\end{pmatrix}
\] is similar to \[
\begin{pmatrix}
B & 0 \\
0 & B
\end{pmatrix}.
\] Prove that \( A \) is similar to \( B \).

5. Determine the Galois group over \( \mathbb{Q} \) of
   a) \((x^3 - 2)(x^2 - 3);\)
   b) \((x^3 - 2)(x^2 + 3).\)

6. Is there any integer \( d \) such that: for any separable extension \( F \) of degree 5 of a field \( K \), if an extension \( E \) of \( K \) is a normal closure of \( F/K \) then \([E : K] \leq d\) ? If yes, prove your bound. If not, justify why.

7. Let \( \xi \in \mathbb{C} \) be such that \( \xi^{2000} = 3 \). Show that \(-3\) is not a sum of squares in \( \mathbb{Q}(\xi) \).

8. Prove the Hilbert Basis Theorem: If \( R \) is a commutative Noetherian ring with identity, then so is \( R[x_1, x_2, \ldots, x_n] \).

9. Let \( A \) be a finite dimensional (not necessarily commutative) \( \mathbb{C} \)-algebra with no zero divisors, but with identity. Show that \( \text{dim}_\mathbb{C} A = 1 \).

10. a) If the ring \( R \) is a principal ideal domain, prove that every nonzero prime ideal in \( R \) is maximal.
  
    b) Is the statement in a) still true if \( R \) is not a principal ideal domain? (Hint: Consider the ring \( \mathbb{Z}[x] \).)

11. Suppose that \( R \) is a ring with identity.
    a) Prove that each free left \( R \)-module is projective.
    b) Prove that a left \( R \)-module \( P \) is projective if and only if each short exact sequence of left \( R \)-modules

\[
0 \to A \to B \to P \to 0
\]

splits.