Please do 7 out of the 11 problems below.

**Question 1.** An algebraic extension field \( F \) of \( K \) is said to be normal over \( K \) if every irreducible polynomial in \( K[x] \) that has a root in \( F \) actually splits in \( F[x] \).

Prove that an algebraic extension \( F \) of \( K \) is normal over \( K \) if and only if for every irreducible \( f \in K[x] \), \( f \) factors in \( F[x] \) as a product of irreducible factors all of which have the same degree.

**Question 2.**
(a). Show that the additive group of rationals \( \mathbb{Q} \) is not free.
(b). Show that the group \( \mathbb{Q}^* \) of all positive rationals (under multiplication) is free abelian with basis \( \{ p : p \) is prime in \( \mathbb{Z} \} \).

**Question 3.** Show that any simple group \( G \) of order 60 is isomorphic to \( A_5 \). (If you want, you may use the fact that for each \( n \geq 2 \), \( A_n \) is the only subgroup of \( S_n \) of index 2.)

**Question 4.** Let \( R \) be an integral domain and for each maximal ideal \( M \), consider the localization \( R_M \) as a subring of the quotient field of \( R \). Show that \( \cap M = R \), where the intersection is taken over all maximal ideals \( M \) of \( R \).

**Question 5.**
(a). Define solvable group.
(b). Prove that any group of order 48 is solvable (without using Burnside's (p,q)-Theorem).

**Question 6.** Let \( R \) be a ring. Show that the following conditions on an \( R \)-module \( P \) are equivalent.

(i). \( P \) is projective.
(ii). Every short exact sequence

\[
0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0
\]

is split exact.
(iii). There is a free module \( F \) and an \( R \)-module \( K \) such that \( F \cong K \oplus P \).

**Question 7.** Let \( C \) and \( D \) be categories, and let \( S \) and \( T \) be covariant functors from \( C \) to \( D \).

(a). Define a natural isomorphism \( \alpha : S \to T \).
(b). If \( B \) is a unitary left module over a ring \( R \) with identity, show that there is a "natural" isomorphism of modules

\[
\alpha_B : R \otimes_R B \cong B.
\]
Question 8. Let $R$ be a commutative ring with identity.
(a). Let $A$ be an ideal in $R$, and assume that $M$ is a finitely generated $R$-module such that $A.M = M$. Show that there is some $a \in A$ satisfying $(1 + a)M = 0$.  
(b). Let $M$ be a finitely-generated $R$-module. Show that $J(R).M = M$ implies $M = \{0\}$. (Here, $J(R)$ denotes the Jacobson radical of $R$.)

Question 9.
(a). Show that if $R$ is a unique factorization domain, then $R$ is integrally closed.
(b). Find the integral closure of $\mathbb{C}[z^3, z^7]$.

Question 10.
(a). Show that $\mathbb{C}[x, y]/(xy - 1)$ (quotient of polynomial ring in 2 variables) is not isomorphic to $\mathbb{C}[t]$ (polynomial ring in 1 variable).
(b). Show that the set \[
\{(m, n) : m, n \in \mathbb{Z}\}
\]
is not an algebraic variety in $\mathbb{C}^2$.

Question 11.
(a). Find the Galois group of $K : \mathbb{Q}$, where $K$ is a splitting field over $\mathbb{Q}$ for $t^4 + t^2 - 6$.
(b). Write down the subgroups of the Galois group.
(c). Write down the corresponding fixed fields.