Answer seven out of the following ten problems. State clearly any results you need.

1. Let $A$ be a commutative Noetherian ring with identity. Show that if $A$ is Noetherian, then so is the power series ring $A[[X]]$.

2. Suppose that $A$ is a commutative ring with identity, $B$ is an $A$-algebra integral over $A$, and $f : A \to K$ is a homomorphism from $A$ to an algebraically closed field $K$. Show that $f$ can be extended to a homomorphism $g : B \to K$.

3. Show that any automorphism of the field $\mathbb{R}$ of real numbers is the identity.

4. Let $p$ be a prime, $n > 1$ an integer, and $G = \text{GL}(n, \mathbb{F}_p)$. Compute the order of $G$, and find a $p$-Sylow subgroup of $G$.

5. Let $p$ be a prime.
   
   (a) Show that if $S_p$ is the symmetric group on $p$ letters, and $H \subseteq S_p$ is a subgroup containing a $p$-cycle and a transposition, then $H = S_p$.
   
   (b) Suppose $f(T) \in \mathbb{Q}[T]$ is irreducible of degree $p$. Show that if $f(T)$ has exactly two non-real roots, then the Galois group of the splitting field of $f(T)$ is $S_p$.

6. Let $M_n(F)$ denote the algebra of $n \times n$ matrices over the field $F$. Fix $p$ with $1 < p < n$ and let $R$ be the subalgebra consisting of those matrices in “block form”

$$
\begin{pmatrix}
A & B \\
0 & C
\end{pmatrix},
$$

where $A$ is a $p \times p$ matrix, and $C$ is an $(n-p) \times (n-p)$ matrix.

   (a) Describe the Jacobson radical of $R$.

   Consider the vector space $V = F^n$ with its natural structure as a left module for $R$, given by matrix multiplication on the left.

   (b) Show that $V$ has an irreducible submodule $W$ of dimension $p$ and that $V/W$ is also irreducible.

   (c) Show that $V$ is indecomposable, i.e. $V$ is not the direct sum of two nonzero $R$-submodules.

7. If $(m,n) = 1$ then compute $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$. Explain.
8. Suppose that $A$ is a commutative ring with identity. If $A^m \cong A^n$ as $A$-modules, show that $m = n$ ($A^m$ denotes the direct sum of $m$ copies of $A$ as an $A$-module, and similarly for $A^n$).

9. Let $A$ be a Noetherian ring and $M$ a finitely generated $A$-module. Denote by $\text{Supp}(M)$ the set of prime ideals $p$ of $A$ such that $M_p \neq 0$, where $M_p$ is the localization of $M$ at $p$. Show that if $M_p = 0$ for all $p$, then $M = 0$.

10. (a) Show that a direct sum of more than one copy of $\mathbb{Z}$ is not generated by a single element.
   (b) Deduce that a direct sum of more than one copy of $\mathbb{Z}$ is not a free group.