Ph.D. Level Examination in Algebra
Fall 1982

Do seven problems, at least two from each section.

I. Group Theory.

1. Let $G$ be the group of all $3 \times 3$ non-singular matrices of determinant 1 with coefficient in the field of 3 elements.
   (a) Find $|G|$.
   (b) Find a Sylow 3-subgroup of $G$.
   (c) Compute $|\text{Syl}_3(G)|$, where $\text{Syl}_3(G)$ is the set of all Sylow 3-subgroups of $G$.

2. Let $G$ be a finite group acting transitively on a set $X$, $x \in X$, and $P$ be a Sylow p-subgroup of $G_x$ for some prime $p$. Consider the set $\text{Fix}(P) = \{y \in X | y^P = y\}$, the set of fixed points of $P$.
   (a) Prove that $N_G(P)$ acts on $\text{Fix}(P)$, i.e., maps $\text{Fix}(P)$ to $\text{Fix}(P)$.
   (b) Prove that $N_G(P)$ acts transitively on $\text{Fix}(P)$.

3. Two permutation representations $G \rightarrow \Sigma(X_1)$, (the symmetric group on $X_1$) are equivalent if there exists $\alpha : X_1 \rightarrow X_2$ such that for all $g \in G$, $x \in X_1$, we have $x(g\pi_1)\alpha = (x\alpha)(g\pi_2)$.
   Prove that the following two permutations of $S_4$ of degree 12 are not equivalent:
   (a) on the right coset space of $<(12)>$.
   (b) on the right coset space of $<(12)(34)>$.

4. Let $G$ be a finite group and $A \leq \text{Aut}(G)$ such that $G = [G,A]$. Suppose $N \triangleleft G$ satisfying $[N,A] = 1$. Use the 3-subgroup lemma to prove $N \triangleleft Z(G)$, the center of $G$.
   (Recall the 3-subgroups lemma: For three subgroups $X$, $Y$, $Z$ of a group $M$ if $[X,Y,Z] = 1$ and $[Y,Z,X] = 1$, then $[Z,X,Y] = 1.$)
II. Homological algebra.

1. Let \( R \) be a ring and consider the following commutative diagram of \( R \)-modules and \( R \)-module homomorphisms such that each row is a short exact sequence.

\[
\begin{array}{cccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0
\end{array}
\]

Prove that if \( \alpha \) and \( \gamma \) are isomorphisms then \( \beta \) is also an isomorphism.

2. Prove the \( \mathbb{Z} \)-module isomorphisms:
   (a) \( \mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/c\mathbb{Z} \)
   (b) \( \text{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/c\mathbb{Z} \)

where \( c = (m,n) \) is the greatest common divisor.

3. A module \( P \) over a ring \( R \) is said to be projective if given any diagram of \( R \)-module homomorphisms with \( g \) surjective

\[
\begin{array}{ccc}
P & \xrightarrow{f} & B \\
\downarrow & & \downarrow g \\
A & \xrightarrow{f} & B
\end{array}
\]

there exists an \( R \)-module homomorphism \( h: P \rightarrow A \) such that the following diagram is commutative

\[
\begin{array}{ccc}
P & \xrightarrow{h} & A \\
\downarrow & & \downarrow \alpha \\
A & \xrightarrow{f} & B
\end{array}
\]

Prove that the following conditions on a ring \( R \) with identity are equivalent:

(a) Every \( R \)-module is projective.

(b) Every short exact sequence of \( R \)-modules splits.
III. Commutative algebra.

1. A non-empty subset $S$ of a ring $R$ with identity is called multiplicative if $ab \in S$ whenever $a, b \in S$.

   (a) If $I$ is a proper ideal of $R$ disjoint from $S$, then prove that there exists an ideal $P$ maximal with respect to containing $I$ and being disjoint from $S$. Furthermore, show that $P$ is a prime ideal.

   (b) If $P$ is a prime ideal of ring $R$, define the localization of $R$ at $P$ and prove that the localization has a unique maximal ideal.

2. Find a reduced primary decomposition for the following ideals. Also give the prime ideals to which the primary ideals belong.

   (a) $(24)$ as an ideal in $\mathbb{Z}$.

   (b) $(2x, x^2)$ as an ideal in $\mathbb{Z}[x]$.

   (c) What theorems allow us to conclude that every ideal in $\mathbb{Z}[x]$ has a primary decomposition.

3. Let $R$ be a unique factorization domain and $K$ its field of fractions.

   (a) Show that if $a \in K$ is integral over $R$ then $a \in R$. (In other words, $R$ is integrally closed.)

   (b) True or false: If $S$ is a subring of $K$ containing $R$, then $S$ is also a unique factorization domain. If true, prove your answer. If false give examples of $R, K$ and $S$ that show it not to be true.

4. Determine, up to isomorphism, all semisimple rings of order $1008 = 2^5 \cdot 3^2 \cdot 7$. 