Ph.D. Examination – Topology
May 2017

Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper.

1. Prove that a compact Hausdorff space is regular.
2. Prove that if \( n > 0 \) every map \( S^n \to S^n \) is homotopic to a map with a fixed point.
3. The oriented surface \( M_g \) of genus \( g \), embedded in \( \mathbb{R}^3 \) in the standard way, bounds a compact region \( R \). Two copies of \( R \), glued together by the identity map between their boundary surfaces \( M_g \), form a closed 3-manifold \( X \). Compute the homology groups of \( X \) using the Mayer–Vietoris sequence.
4. Let \( f : S^1 \to \mathbb{R} \) be a continuous map. Prove that there exists a point \( x \in S^1 \) with \( f(x) = f(-x) \). (Note: do not simply state the Borsuk–Ulam Theorem; give a direct proof.)
5. Let \( M \) be a closed simply-connected orientable 3-manifold. Compute the integral homology and cohomology of \( M \). What can you say about \( \pi_i(M), i \leq 3 \)?

Answer the following with complete definitions, statements, or short proofs.

6. Prove that for a finite CW-complex \( X \), \( H^1(X; \mathbb{Z}) \) is torsion-free.
7. Compute \( \chi(\mathbb{C}P^3 \times \mathbb{R}P^2 \times S^2) \)
8. Give an example of a space that is path-connected but not locally path-connected.
9. State the Urysohn Lemma.
10. Prove that if \( m \neq n \), then \( \mathbb{R}^m \) is not homeomorphic to \( \mathbb{R}^n \).
11. Does the function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = \arctan x \) admit a continuous extension \( \bar{f} : \beta\mathbb{R} \to \mathbb{R} \) to the Stone–Čech compactification? What about the function \( g(x) = e^x \)?
12. State the Lefschetz Fixed Point Theorem.
13. Describe all the connected covering spaces \( E \to S^1 \).
14. Does the following exact sequence of abelian groups necessarily split? Prove or give a counterexample.
\[
0 \to \mathbb{Z} \to A \to B \to 0
\]
15. Compute the integral homology of the space \( \mathbb{R}P^4 \times \mathbb{C}P^2 \).