1. (a) Show the matrix norm equality for $A \in \mathbb{C}^{m \times n}$

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$$

(b) Explain why the matrix 1-norm, 2-norm and $\infty$-norm are the most commonly used of the matrix $p$-norms in scientific computing.

(c) Show $\rho(A) \leq \|A\|$ where $\|A\|$ is any subordinate (induced) matrix norm and $\rho(A)$ is the spectral radius of $A$.

2. Let $A \in \mathbb{C}^{m \times n}$ with $\text{rank}(A) = n < m$. Let $A = QR$ be the QR decomposition of $A$, and $A = Q_1R_1$ be the economy QR decomposition.

(a) Show $Q_1Q_1^*$ is an orthogonal projector onto $\text{Col}(A)$.

(b) Let $b \in \mathbb{C}^m$. Write down an expression for the least-squares solution to $Ax = b$ as the solution to an $n \times n$ system in terms of $Q_1$, (and/or $Q_1^*$), $R_1$, $x$ and $b$.

3. Let $A = U\Sigma V^*$ be the singular value decomposition of $A \in \mathbb{C}^{m \times n}$ with $\text{rank}(A) = p$ and $p \leq n \leq m$.

(a) Show $\text{Col}(A^*) = \text{Span}\{v_1, v_2, \ldots, v_p\}$, where $v_1, \ldots, v_p$ are the first $p$ columns of $V$.

(b) Show $\text{Null}(A) = \text{Span}\{v_{p+1}, v_{p+2}, \ldots, v_n\}$.

(c) Suppose the right singular vectors $v_1, \ldots, v_p$ have been computed. Describe how to compute the left singular vectors $u_1, \ldots, u_p$ (without solving a spectral problem).

4. Let $\|\cdot\|$ be a subordinate (induced) matrix norm. If $A$ is $n \times n$ invertible and $E$ is $n \times n$ with $\|A^{-1}\||E| < 1$, then show

(a) $A + E$ is nonsingular

(b) 

$$\|(A + E)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\||E|}.$$  

5. Consider the matrix $A$ given by

$$
\begin{pmatrix}
1 & -1 & 2 & 0 \\
-1 & 4 & -1 & 1 \\
2 & -1 & 6 & -2 \\
0 & 1 & -2 & 4
\end{pmatrix}
$$

Suppose the eigenvalues of $A$ are all distinct (they are) and satisfy $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$.

(a) Show that $A$ is positive definite.

(b) Describe an algorithm that could be used to converge to $\lambda_4$.

(c) Describe an algorithm that could be used to converge to $\lambda_2$. 


Numerical Analysis Exam: May, 2019

Do 4 (four) problems.

1. Consider the fixed point problem \( x = f(x) \) where \( f(x) = e^{-(2+x)} \).
   (a) Find the largest open interval in \( \mathbb{R} \) where \( f(x) \) is a contraction.
   (b) Assuming all computations are done in exact arithmetic, find the largest open interval in \( \mathbb{R} \) where the fixed-point iteration \( x_{k+1} = f(x) \) is assured to converge.
   (c) Write a Newton iteration for finding the fixed-point.

2. Let \( x_1, x_2, \ldots, x_{n+1} \) be \( n + 1 \) distinct numbers. Let \( l_j(x) \) be the associated Lagrange basis polynomials, \( j = 1, \ldots, n + 1 \).
   (a) State the definition of \( l_j(x) \) and show that \( \{l_j(x)\}_{j=1}^{n+1} \) form a basis for \( \mathcal{P}_n \), the space of polynomials of degree at most \( n \).
   (b) Show that
   \[
   \sum_{j=1}^{n+1} (x - x_j)^k l_j(x) = 0, \quad \text{for all } k = 1, \ldots, n.
   \]

3. Consider the interval \( [a, b] \) with the partition \( a = x_1 < x_2 < \cdots < x_n < x_{n+1} = b \). Suppose \( s(x) \) is the natural cubic spline that interpolates the data \( \{(x_i, y_i)\}_{i=1}^{n+1} \) and that \( g \in C^2[a, b] \) interpolates the same data. Show that
   \[
   \int_a^b (s''(x))^2 \, dx \leq \int_a^b (g''(x))^2 \, dx.
   \]

4. (a) Consider the inner product on \( C(0, 2) \) given by \( (f, g) = \int_0^2 f(t) g(t) \, dt \). Find three orthonormal polynomials \( \phi_0, \phi_1, \phi_2 \) on \( (0, 2) \) with respect to the given inner product such that the degree of \( \phi_n \) is equal to \( n \), \( n = 0, 1, 2 \).
   (b) Find the nodes \( t_1 \) and \( t_2 \) and weights \( w_1 \) and \( w_2 \) which yield the weighted Gaussian Quadrature formula
   \[
   \int_0^2 f(t) \, dt \approx w_1 f(t_1) + w_2 f(t_2)
   \]
   with degree of exactness \( m = 3 \). You should find the nodes exactly, and may leave the weights \( w_1, w_2 \) in integral form.

5. Let \( f \in C^\infty(a-H, a+H) \), and let \( h < H \). Let \( x_0 = a - h \), \( x_1 = a \) and \( x_2 = a + h \).
   (a) Find the finite difference approximation to \( f'(a) \) based the interpolant \( p_2 \) which satisfies \( p_2(x_0) = f(x_0), p_2(x_1) = f(x_1) \) and \( p_2(x_2) = f(x_2) \).
   (b) Let \( \psi_0(h) = \psi(h) \) be the difference approximation to \( f'(a) \) found in part (a). Assume (in exact arithmetic) \( \psi(h) \to \psi(0) = f'(a) \) as \( h \to 0 \), and that \( \psi(h) \) has the asymptotic expansion
   \[
   \psi(h) = \psi(0) + a_2 h^2 + a_4 h^4 + a_6 h^6 + \ldots.
   \]
   Find the general Richardson extrapolation formula for \( \psi_k(h) \) based on \( \psi_{k-1}(h/2) \) and \( \psi_{k-1}(h) \).