1. Let $Q$ denote an $n$ by $n$ symmetric matrix and let $d_k$, $1 \leq k \leq n$, denote a collection of directions that are $Q$-conjugate.

(a) Suppose the $Q$ is positive definite. Show that the directions $d_k$, $1 \leq k \leq n$, are linearly independent, and express the solution of the linear system $Qx = b$ in terms of the conjugate directions.

(b) If $d_k^T Q d_k > 0$ for each $k$, then show that $Q$ is positive definite.

2. Consider the following variational problem:

$$\min \left\{ \int_0^1 \frac{1}{2} u'(x)^2 + e^{u(x)} \, dx : u \in H^1([0,1]), \quad u(0) = u(1) = 0 \right\}$$

(a) What is the first-order necessary optimality condition (Euler equation) for this variational problem?

(b) Consider a uniform mesh $x_k = kh$ where $h = 1/N$; let $u_k$ denote an approximation to $u(x_k)$. Of course, $u_0 = u_N = 0$. Give a finite difference approximation in terms $u_1, \ldots, u_{N-1}$ to the solution of the Euler equation.

(c) Let $F(u)$ denote the finite difference system where $u = (u_1, u_2, \ldots, u_{N-1}) \in \mathbb{R}^{N-1}$, and let $u^*$ denote the vector formed by evaluating the solution of the Euler equation at the mesh points $x_1, x_2, \ldots, x_{N-1}$. Obtain a bound for the components of $F(u^*)$ in terms of the mesh spacing $h$.

3. Consider the following variational problem where $f$ is a given smooth function:

$$\min \left\{ \Phi(u) := \int_0^1 \frac{1}{2} u'(x)^2 + f(x)u(x) \, dx : u \in H^1([0,1]), \quad u(0) = u(1) = 0 \right\}.$$

Let $S^h$ denote the piecewise linear finite element space defined on a uniform mesh with $v^h(0) = v^h(1) = 0$ for all $v^h \in S^h$. Obtain a bound for the error in the finite element approximation $u^h$ obtained by minimizing $\Phi$ over $S^h$.

4. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable, $x^* \in \mathbb{R}^n$, $\nabla f(x^*) = 0$, and $\nabla^2 f(x^*)$ is positive definite. Show that $x^*$ is a strict local minimizer of $f$.

5. Suppose that a quadratic $\Phi(x) = \frac{1}{2} x^T Q x - b^T x$ is minimized by steepest descent: $x_{k+1} = x_k - s_k g_k$, where $g_k = \nabla \Phi(x_k)$ and $s_k$ is the stepsize.

(a) Derive the formula for the Cauchy step.

(b) Derive the formula for the BB step.
6. Consider the system of differential equations \( \dot{x}(t) = f(x(t), u(t)), \) \( x(0) = x_0, \) where \( f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \) and \( f \) satisfies the global Lipschitz condition

\[
\|f(x_1, u_1) - f(x_2, u_2)\| \leq L(\|x_1 - x_2\| + \|u_1 - u_2\|).
\]

Show that the differential equation has a global solution and on any interval \([0, T]\), the following bound holds for all \((u_1, u_2) \in L^{\infty}([0, T])\):

\[
\|x_1 - x_2\|_{L^{\infty}} \leq e^{LT} \|u_1 - u_2\|_{L^1},
\]

where \(x_i\) is the solution of the differential equation associated with \(u_i, i = 1, 2\).

7. Suppose that \(P\) and \(Q\) are continuous on \([0, 1]\) and the following functional is nonnegative over all \(h \in H^1_0([0, 1])\):

\[
\Omega(h) = \int_0^1 P(x)h'(x)^2 + Q(x)h(x)^2 \, dx.
\]

Show that \(P \geq 0\) on \([0, 1]\).

8. Suppose that \(u: [0, 1] \rightarrow \mathbb{R}\) is twice continuously differentiable, and let \(u^I\) be the continuous, piecewise linear interpolant of \(u\). Prove the following bound:

\[
\|u - u^I\|_{L^{\infty}} \leq \frac{h^2}{8} \|u''\|_{L^{\infty}}.
\]

9. Suppose that \(f: \mathbb{R}^n \rightarrow \mathbb{R}\) is a convex function.

(a) If \(f\) is continuously differentiable, then show that for all \(x, y \in \mathbb{R}^n\), we have

\[
f(y) \geq f(x) + \nabla f(x)(y - x).
\]

(b) If \(f\) is twice continuously differentiable, then show that the Hessian \(\nabla^2 f(x)\) is positive semidefinite for all \(x \in \mathbb{R}^n\).

10. Suppose that \(\Phi: \mathbb{R}^n \rightarrow \mathbb{R}^n\) and \(x^* \in \mathbb{R}^n\) is a fixed point of \(\Phi\).

(a) If \(\Phi\) is a contraction mapping with contraction constant \(\lambda\), then show that the iteration \(x_{k+1} = \Phi(x_k)\) converges to \(x^*\) from any starting guess \(x_0\), and

\[
\|x_k - x^*\| \leq \lambda^k \|x_0 - x^*\|.
\]

(b) Suppose that \(\Phi\) is continuously differentiable on a convex set \(\mathcal{K} \subset \mathbb{R}^n\). Let \(\mu\) denote the supremum of the singular values of \(\nabla \Phi\) over \(\mathcal{K}\). Show that

\[
\|\Phi(x) - \Phi(y)\| \leq \mu\|x - y\|
\]

for all \(x, y \in \mathcal{K}\).