1. Consider the set $T_n$ of (not rooted) trees with $n \geq 3$ labeled leaves for which each interior (non-leaf) vertex has degree 3.

(a) Prove that each tree in $T_n$ has exactly $2n - 3$ edges.

(b) Prove that the number of trees in $T_n$ is $(2n-5)!! = 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-7) \cdot (2n-5)$.
2. The famous Dilworth Theorem for a finite poset $P$ states that the minimum number of chains in any partition of $P$ into chains is equal to the maximum number of elements in an antichain of $P$.

(a) Prove that if $P$ has size at least $rs + 1$, then there is either a chain of size $r$ or an antichain of size $s$.

(b) There are many proofs of the following Erdős-Szekeres theorem: For any sequence $a_1, a_2, \ldots, a_{n^2+1}$ of integers, there is a subsequence of length $n + 1$ that is monotone. Prove the Erdős-Szekeres theorem using Dilworth’s theorem, i.e., part (a).
3. Prove that

\[ \sum_{i=0}^{n} (-1)^i \binom{n}{i} i^k = \begin{cases} 0 & \text{if } 0 \leq k < n \\ (-1)^n n! & \text{if } k = n. \end{cases} \]

What is the sum if \( k > n \)?

Hint: Count surjective mappings from \([k]\) to \([n]\).
4. (a) Prove that a planar graph of girth (minimum cycle length) at least 6 has a vertex of degree 2.

(b) The Four Color Theorem states that any planar graph is 4-colorable. Grotzsch proved that any triangle-free planar graph is 3-colorable. Without using his result, prove that a planar graph of girth at least 6 is 3-colorable.

Recall: The *girth* of a graph is the length of the shortest cycle (and $\infty$ if the graph has no cycles).
5. How many strings can be formed using the alphabet \{A, B, C, D, E\} if
   
   (a) the letter \(A\) occurs an odd number of times

   (b) the letters \(A\) and \(B\) are both used an odd number of times.
6. Let \( f(n) \) be the number of ways to tile a \( 1 \times n \) path with \( 1 \times 2 \) tiles that are red, blue, or green, and \( 1 \times 1 \) tiles that are yellow, orange, black, or white.

(a) Find an explicit formula for \( f(n) \).

(b) On average, how many \( 1 \times 1 \) tiles will a \( 1 \times n \) path contain?
7. Choose a derangement $p$ of length $n$ uniformly at random. Recall that a derangement is a permutation with no 1-cycles. On average, how many 2-cycles does $p$ contain?
8. Let $C$ be a binary code of length $n$ and minimum distance $d \geq 2e + 1$. Prove the following two bounds (the Hamming bound and Singleton bound).

$$|C| \leq 2^n / \sum_{i=0}^{e} \binom{n}{i}$$

$$|C| \leq 2^{n-d+1}$$