ANALYSIS QUALIFYING EXAM MAY 2019

Do six problems.

1. Let $X, Y$ be topological spaces. Prove that if $f : X \to Y$ is continuous, then $f$ is Borel measurable.

2. Rigorously determine the following expressions:
   (a) $\lim_{n \to \infty} \int_0^\infty \frac{\sin \frac{x}{n}}{(1 + \frac{x}{n})^n} dx$
   (b) $\lim_{n \to \infty} \int_0^\infty \frac{n \sin \frac{x}{n}}{x(1 + x^2)} dx$
   (c) $\lim_{n \to \infty} \int_0^\infty \frac{n}{n^2 + x^2} dx$

3. Prove that if a sequence of functions converges in $L^1$, then it has a subsequence that converges pointwise almost everywhere.

4. State and prove the Hahn decomposition theorem.

5. Let $X$ be a Banach space. Let $Y, Z \subseteq X$ be closed subspaces such that $Y + Z = X$ and $Y \cap Z = \{0\}$. Prove there is $C > 0$ such that for all $y \in Y$ and $z \in Z$,
   $C(\|y\| + \|z\|) \leq \|y + z\| \leq \|y\| + \|z\|.$

6. Let $X$ be a Banach space. Show that the natural map from $X$ to $X^{**}$ is injective.

7. State and prove the Cauchy-Schwarz inequality over a complex Hilbert space.

8. Prove or give a counterexample: If $f, g$ are smooth functions on $\mathbb{R}$ such that $f, g \in L^1(\mathbb{R})$, then $f * g$ is a smooth function and $f * g \in L^1(\mathbb{R})$.

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1For the proof, you may want to consider a sequence of sets such that $\mu(E_n) \to \sup_{E} \mu(E)$ rapidly, under the assumption that such a supremum is bounded.