(1) Prove that if $X, Y$ are topological spaces and $f : X \to Y$ is continuous, then $f$ is Borel measurable.

(2) (Do three of four.)
(a) Give an example, if possible, of a closed set $E \subset \mathbb{R}$ of positive Lebesgue measure that contains no interval.
(b) Give an example, if possible, of a sequence $f_n : [0, 1] \to \mathbb{R}$ of Lebesgue measurable functions that converges in measure, but not pointwise a.e.
(c) Give an example, if possible, of a premeasure on a Boolean algebra $\mathcal{A}$ without a unique extension to a measure on the $\sigma$-algebra generated by $\mathcal{A}$.
(d) Determine the limit of the sequence $(a_n)_{n=1}^{\infty}$ defined by
\[
a_n = \int_1^{\infty} \frac{\cos^2(\pi t)}{t^n} \, dt.
\]

(3) Suppose $(X, \mathcal{M}, \mu)$ is a $\sigma$-finite measure space and $\mathcal{N} \subset \mathcal{M}$ is a (sub) $\sigma$-algebra. Let $\nu = \mu|_{\mathcal{N}}$. Show, if $(X, \mathcal{N}, \nu)$ is a $\sigma$-finite measure space and $f \in L^1(\mu)$, then there exists a $g \in L^1(\nu)$ such that, for all $E \in \mathcal{N}$,
\[
\int_E g \, d\nu = \int_E f \, d\mu.
\]
Is the $\sigma$-finiteness hypothesis on $\nu$ necessary?

(4) Given $f \in L^1([0, 1])$, let $g(x) = \int_1^x \frac{f(t)}{t} \, dt$. Show $g \in L^1([0, 1])$ and
\[
\int_0^1 g = \int_0^1 f.
\]

(5) (Do three of four)
(a) Is there a norm on the vector space $c_{00} = \{f : \mathbb{N} \to \mathbb{C} : \text{there exists an } N \text{ such that } f(n) = 0 \text{ for } n \geq N\}$ that makes $c_{00}$ a Banach space?
(b) Explain what is meant by the Fourier transform of a function $f \in L^2(\mathbb{R})$.
(c) Is there bounded linear bijection between $C([0, 1])$ and $L^\infty([0, 1])$?
(d) Give an example, if possible, of a sequence of unit vectors $(f_n)$ from $L^2(\mathbb{R})$ that converges weakly to some $f \in L^2(\mathbb{R})$, but not in norm.

(6) Show, if $E \subset \mathbb{R}$ is a Lebesgue measurable set of positive measure, then $E - E = \{x - y : x, y \in E\}$ contains an open interval.
(7) Suppose \( X \) is a Banach space and \( \mathcal{M} \) and \( \mathcal{N} \) are closed subspaces. Show, if for each \( x \in X \) there exist unique \( m \in \mathcal{M} \) and \( n \in \mathcal{N} \) such that

\[
x = m + n,
\]

then the mapping \( P : X \to \mathcal{M} \) defined by \( Px = m \) is bounded.

(8) Suppose \((X, \mathcal{M}, \mu)\) is a \( \sigma \)-finite measure space and \( 1 \leq q < p < \infty \). Show, \( L^q \subset L^p \) if and only if there exists a \( \delta > 0 \) such that if \( E \in \mathcal{M} \) then either \( \mu(E) = 0 \) or \( \mu(E) \geq \delta \).