First-Year Second-Semester Exam
May 2018

Answer the following problems and show all your work. Support all statements to the best of your ability. Use a separate sheet of paper for each problem. Each is worth 10 points.

1. Let $X$ be a completely regular space.
   (a) Define a compactification of $X$.
   (b) Define $\beta(X)$, a Stone-Čech compactification of $X$, in terms of its universal property.
   (c) Show that if $\beta(X)$ exists then it is unique up to homeomorphism.

2. Suppose that the continuous maps $f, g : X \to Y$ are homotopic and that the continuous maps $h, k : Y \to Z$ are homotopic. Show that $h \circ f$ is homotopic to $k \circ g$.

3. Show that the fundamental group of the circle is isomorphic the additive group of integers. That is, $\pi_1(S^1, 1) \cong \mathbb{Z}$.

4. State and prove the Brouwer fixed point theorem for the unit disk, $D^2$.

5. Prove that for $n \geq 2$, the fundamental group of the $n$-sphere, $S^n$, is 0.

Answer the following with complete definitions or statements or short proofs. Each is worth 5 points.

6. State the Tietze Extension Theorem.

7. Define a Baire space. State the Baire category theorem.

8. Suppose that $f, g : X \to \mathbb{R}^n$ are continuous functions. Show that $f$ and $g$ are homotopic.

9. Define the retraction of a topological space $X$ onto a subspace $A$. If $j : A \hookrightarrow X$ is the inclusion of a retract, then show that the induced map on fundamental groups $j_\#$ is injective.

10. Show that $S^1$ is not homeomorphic to $S^2$.

11. Show that $\mathbb{R}^2$ is not homeomorphic to $S^2$.

12. Is the squaring map $s : S^1 \to S^1$, $s(z) = z^2$, nulhomotopic?

13. What is the fundamental group of the torus $T^2 = S^1 \times S^1$?

14. State the Seifert-van Kampen Theorem.

15. Describe a quotient of a polygon whose fundamental group is isomorphic to $\mathbb{Z}/3$. 