First-year Analysis Examination  
Part One  
January 2018

Answer FOUR questions in detail. 
State carefully any results used without proof.

1. Let \((a_n : n \geq 0)\) be a bounded real sequence. For each \(N \geq 0\) define \(\bar{a}_N = \sup\{a_n : n \geq N\}\) and let \(s = \inf\{\bar{a}_N : N \geq 0\}\). Prove:
   (i) if \(s < t\) then \(a_n < t\) eventually, in the sense \((\exists N)(\forall n \geq N)(a_n < t)\);
   (ii) if \(s > r\) then \(a_n > r\) frequently, in the sense \((\forall N)(\exists n \geq N)(a_n > r)\).

2. Let \(X\) be a metric space. Assume that there exists \(\delta > 0\) such that the distance between distinct points of \(X\) is always \(\delta\) or greater. Determine precisely which subsets of \(X\) are open, which closed, which compact and which connected.

3. Let \(f : X \to \mathbb{R}\) have the property that if \(a\) is any real number then \(U(a) = \{x : f(x) < a\}\) is an open subset of the compact metric space \(X\).
   (i) Prove that \(f\) is bounded above on \(X\).
   (ii) Prove that \(s = \sup\{f(x) : x \in X\}\) is a value of \(f\).
   For (ii): suppose \(s\) not a value and consider the function \(F = 1/(s - f)\).

4. Let \(f : X \to Y\) be a bijection between metric spaces; assume that \(Y\) is complete and that \(f^{-1}\) is continuous.
   (i) Show that if \(f\) is uniformly continuous then \(X\) is complete.
   (ii) Show that if \(f\) is only continuous then \(X\) can fail to be complete.

5. Let \(f : (a, b) \to \mathbb{R}\) be differentiable and assume that its derivative \(f'\) is bounded. Prove that the right-hand limit \(\lim_{t \to a^+} f(t)\) exists.