1. Prove that every square matrix $A$ has a Schur factorization.

2. Given a matrix $A \in \mathbb{C}^{m \times n}$, let

$$B = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \quad \text{and} \quad C = A^*A$$

(a) Show that the singular values of $A$ are the absolute values of eigenvalues of $B$.

(b) Show that the singular values of $A$ are the square roots of eigenvalues of $C$.

(c) Assume now that $A$ is square and invertible. Compute the two-norm condition numbers of $B$ and $C$ in terms of the two-norm condition number of $A$.

(d) If $A$ has condition number bigger than one, which has the larger condition number, $B$ or $C$?

3. (a) Compute $\det(\lambda I + uv^*)$ when $\lambda \in \mathbb{C}$, $I$ is the $m \times m$ identity matrix, and $u, v \in \mathbb{C}^m$.

(b) Prove necessary and sufficient conditions for $I + uv^*$ to be nonsingular and when it is, give a formula for its inverse.

4. (a) Given Cholesky decomposition of the Hermitian positive definite matrix $A = R^*R$, prove that $\|R\|_2 = \|R^*\|_2 = \|A\|_2^{1/2}$.

(b) Now assume that $B$ is a matrix that can be expressed as $B = T^*T$ for some upper triangular matrix $T$. Show that $B$ is Hermitian and positive semi-definite, i.e. $x^*Bx \geq 0$ for all $x$.

5. Let $\{q_1, q_2, \ldots, q_n\}$ be an orthonormal subset of $\mathbb{C}^m$. Show that

$$P = \sum_{i=1}^n q_iq_i^*$$

is an orthogonal projector with range equal to the span of $\{q_1, q_2, \ldots, q_n\}$.