Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps. Make clear to us that you completely understand why your claims are true. Hand in solutions to 7 out of the 8 problems. If you hand in more than 7, then the first 7 will be graded.

1) Let \((M, d)\) be a metric space. Show that a metric \(D\) is defined on \(M\) by

\[
D(x, y) = \frac{d(x, y)}{1 + d(x, y)}.
\]

Show also that \(d\) and \(D\) determine precisely the same collection of open sets in \(M\).

2) Let \(\{a_n\}_{n=1}^{\infty}\) be a sequence of real numbers and let \(\pi\) be an arbitrary permutation on the set of natural numbers. Consider the following statements:

(i) if \(\{a_n\}_{n=1}^{\infty}\) converges, then so does \(\{a_{\pi(n)}\}_{n=1}^{\infty}\);

(ii) if \(\sum_{n=1}^{\infty} a_n\) converges, then so does \(\sum_{n=1}^{\infty} a_{\pi(n)}\).

For each statement, provide a proof (if true) or a counterexample (if false).

3) Let \(\mathcal{F}\) be a \(\sigma\)-algebra of subsets of the set \(\Omega\) and let \(\{f_n\}_{n=1}^{\infty}\) be a sequence of \(\mathcal{F}\)-measurable real-valued functions on \(\Omega\). Prove that each of the following subsets of \(\Omega\) lies in \(\mathcal{F}\):

(a) \(R = \{\omega \in \Omega : \{f_n(\omega)\}_{n=1}^{\infty} \text{ converges to a real number} \}\);

(b) \(Q = \{\omega \in \Omega : \{f_n(\omega)\}_{n=1}^{\infty} \text{ converges to a rational number} \}\);

(c) \(P = \{\omega \in \Omega : \{f_n(\omega)\}_{n=1}^{\infty} \text{ converges to an irrational number} \}\).

4) Suppose \(f : [0, 1] \to \mathbb{R}\) is Riemann integrable. Prove that if \(f\) takes nonnegative values, \(f(0) > 0\) and \(f\) is continuous at 0, then \(\int_0^1 f(x) \, dx > 0\).

5) Suppose \(f : [0, 1] \to \mathbb{R}\). Prove that if \(f\) is increasing and its range is an interval, then \(f\) is continuous.

6) Suppose \(f : [0, 1] \to \mathbb{R}\) is differentiable. Prove that if \(f'\) is increasing, then \(f'\) is continuous.

7) Suppose that \(\{f_n\}\) and \(\{g_n\}\) are sequences of bounded real-valued functions which converge uniformly on a set \(E\). Prove that \(\{f_n g_n\}\) converges uniformly on \(E\). Give an example which illustrates that the conclusion can be false if the condition of boundedness is dropped.

8) Let \((\Omega, \Sigma, \mu)\) be a measure space with \(\mu(\Omega) < \infty\). Show that if \(\{f_n\}\) is a sequence of bounded, measurable, real-valued functions on \(\Omega\) and \(f_n \to f\) uniformly on \(\Omega\), then

\[
\lim_{n \to \infty} \int_{\Omega} f_n \, d\mu = \int_{\Omega} f \, d\mu.
\]

Show with a suitable counterexample that if the assumption \(\mu(\Omega) < \infty\) is dropped, then the conclusion can be false.