First Year Examination in Analysis September 2011

To obtain credit all work must be presented in a neat and logical fashion, without gaps, reasons given in your argument.

(1) Let $A$ be a subset of a metric space. Prove that the set of limit points of $A$ is a closed set.

(2) Let $f$ be a bounded real valued function defined on the reals $\mathbb{R}$. For $T \subset \mathbb{R}$ let

$$A(T) = \sup \{f(x) - f(y) : x, y \in T\}.$$ 

Let $\omega(x) = \lim_{h \to 0^+} A(T_h)$ where $T_h = \{y : |y - x| < h\}$. Fix $x \in \mathbb{R}$. Does $\omega(x)$ exist? If $\omega(x) = 0$, is $f$ continuous at $x$? If $f$ is continuous at $x$, is $\omega(x) = 0$? Prove or disprove.

(3) Let $f$ be a continuous real valued function defined on a closed bounded interval. Using the fact that the interval is compact, prove $f$ is uniformly continuous on the interval.

(4) Let $f : [a, b] \to \mathbb{R}$ be given such that $f$ is continuous on $[a, b]$ and differentiable on $(a, b)$ and $f(a) = f(b)$. Prove that there exists an $x \in (a, b)$ such that $f'(x) = 0$. (Do not use the MVT, only properties of continuity and differentiability.)

(5) Suppose $f$ is real valued and continuous on $[0, \infty)$ such that $f'$ exists on $(0, \infty)$. Further assume that $f(0) = 0$ and $f'$ is monotonically increasing on $(0, \infty)$. Let $g(x) = \frac{f(x)}{x}$ on $(0, \infty)$. Prove $g$ is monotonically increasing on $(0, \infty)$. [Hint: Examine $g'$].

(6) Given $\sum a_n x^n$, $a_n$, $x$ real. Let $R$ be the radius of convergence, $0 < R < \infty$. Suppose that $I$ is a closed and bounded interval contained in $(-R, R)$. Does $\sum a_n x^n$ converge uniformly on $I$? Prove.

(7) State the Stone-Weierstrass theorem. Outline the main steps of the proof of the theorem giving as many details as you can.

(8) Let $A$ be a Lebesgue measurable subset of a bounded interval. Let $\epsilon > 0$ be given. Prove that there exists an open set $B$ containing $A$ such that $m(B - A) < \epsilon$, where $m$ is Lebesgue measure.

(9) Let $\{f_n\}$ be a sequence of Lebesgue integrable functions defined on $\mathbb{R}$ which converges uniformly to $0$ on $\mathbb{R}$. Does $\lim_{n} \int_{\mathbb{R}} f_n \, dm = 0$? Prove or disprove.