First-Year Analysis Examination  
May 2011

Do SEVEN of the eight problems.

Print your name on each page. Solutions must be written in a neat, logical and professional fashion in order to receive credit. State carefully any substantial theorems used.

1. (a) Show that every closed subset of the real line is a countable union of compact sets.
(b) Is this true in an arbitrary metric space? Prove.

2. Let $A$ and $B$ be disjoint nonempty closed sets in a metric space $X$. Show that there exists a continuous function $f : X \to [0, 1]$ such that $f$ takes value 0 on $A$ and 1 on $B$. Hint: Consider $f(p) = \operatorname{dist}(p, A)/[\operatorname{dist}(p, A) + \operatorname{dist}(p, B)]$.

3. State and prove the theorem concerning the uniform convergence of a monotone decreasing sequence of continuous functions on a compact metric space that converges pointwise to a continuous function.

4. Let $f : X \to Y$ be continuous, where $X$ and $Y$ are metric spaces and $X$ is compact. Prove that $f$ is uniformly continuous on $X$ by proceeding as follows: if $f$ is NOT uniformly continuous then there exist an $\varepsilon > 0$ and sequences $\{p_n\}$ and $\{q_n\}$ in $X$ such that $d_X(p_n, q_n) \to 0$, but $d_Y(f(p_n), f(q_n)) > \varepsilon$. Complete all details.

5. (a) State Fatou’s Theorem and Lebesgue’s Dominated Convergence Theorem (LDCT).
(b) Use Fatou’s Theorem to prove LDCT.

6. Let $\{f_n\}$ be a sequence of integrable functions on a measure space $(X, \Sigma, \mu)$ such that $\int_X |f_n| d\mu < 1/2^n$ for each $n$. Discuss and prove the pointwise convergence of $\sum_n f_n(x)$ for $x \in X$.

7. Let $f$ be an integrable function on the measure space $(X, \Sigma, \mu)$. Let $\varepsilon > 0$. Prove that there exists a $\delta > 0$ such that $|\int_A f d\mu| < \varepsilon$ whenever $\mu(A) < \delta$, $A \in \Sigma$. Hint: Approximate $|f|$ by a simple integrable function $s$. 

1
8. Let $f$ be Lebesgue integrable on $[0, 1]$. Suppose $\int_a^b f\,dx = 0$ for all $0 \leq a \leq b \leq 1$. Show $\int_A f\,dx = 0$ for every measurable subset $A$ of $[0, 1]$. Hint: What is the value of the integral on an open subset of $[0, 1]$? Then apply the result in problem 7.