First-Year Analysis Examination
September 2010

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Let \((a_n)\) and \((b_n)\) be bounded sequences of strictly positive numbers. Prove that
\[
\limsup(a_nb_n) \leq \limsup(a_n)\limsup(b_n).
\]
Can this inequality be strict?

2. Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces; give their product \(X \times Y\) the metric \(d\) defined by
\[
d((x_1,y_1),(x_2,y_2)) = d_X(x_1,x_2) + d_Y(y_1,y_2).
\]
Prove that:
(i) if \(X\) and \(Y\) are compact then so is \(X \times Y\);
(ii) if \(X\) and \(Y\) are complete then so is \(X \times Y\).

3. The real-valued function \(f\) is continuous on \([a,b]\) and differentiable on \((a,b)\). Show that if \(f(a) = f(b) = 0\) then there exists \(t\) in \((a,b)\) such that \(f'(t) = f(t)\).
Suggestion: Consider \(\phi(t)f(t)\) for suitable \(\phi\).

4. Show that if \(f : [0,1] \to \mathbb{R}\) is continuous then
\[
\lim_{n \to \infty} \int_0^1 f(t^n)dt = f(0).
\]

5. Prove Dini's Theorem: that if \((f_n)\) is a sequence of continuous functions decreasing pointwise to zero on a compact space, then the convergence is uniform.

6. Let \(f\) be a continuous real-valued function on the unit square \([0,1] \times [0,1]\). Prove, by the Stone-Weierstrass Theorem or otherwise, that
\[
\int_0^1 \left(\int_0^1 f(x,y)dx\right)dy = \int_0^1 \left(\int_0^1 f(x,y)dy\right)dx.
\]

7. Prove that if \((f_n)\) is a pointwise-bounded sequence of measurable functions then \(\sup f_n\) is measurable; prove also that if \((f_n)\) converges pointwise then \(\lim f_n\) is measurable.

9. Let the function $f : \mathbb{R} \to \mathbb{R}$ have arbitrarily small positive periods, in the sense that if $\delta > 0$ then there exists $T \in (0, \delta)$ such that $f(t + T) = f(t)$ for all $t \in \mathbb{R}$.

(i) Prove that if $f$ is continuous then $f$ is constant.
(ii) What if $f$ is not assumed to be continuous?

Suggestion: Show that if $0 < a < b$ then there exist a period $T$ and an integer $n$ such that $a < nT < b$. 