First-Year Analysis Examination
September 2006

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Suppose that $S_1$ and $S_2$ are nonempty bounded subsets of $\mathbb{R}$. Prove that

$$\text{lub}\{x_1 + x_2 : x_1 \in S_1, x_2 \in S_2\} = \text{lub}S_1 + \text{lub}S_2$$

where lub denotes least upper bound.

2. Let $f$ be a continuous, one-to-one mapping of a compact metric space $X$ onto a metric space $Y$. Prove that $f^{-1}$ is a continuous mapping from $Y$ to $X$. Show by example that if compactness is not assumed then the inverse may not be continuous.

3. Let $(s_n)_{n=1}^{\infty}$ be a sequence of complex numbers and let $t_n = (s_1 + \cdots + s_n)/n$. Prove that $s_n \to s$ implies $t_n \to s$. Show that the converse is false, by a suitable example.

4. Suppose that $f$ is a differentiable real-valued function on $[a, b]$ and suppose that $f''(a) < \lambda < f''(b)$. Prove that there exists a point $x$ in $(a, b)$ such that $f'(x) = \lambda$.

5. Let $E$ be a compact metric space and let $(f_n)$ be a sequence of continuous real-valued functions on $E$ converging pointwise to a function $f$ on $E$; suppose that $f_n(x) \geq f_{n+1}(x)$ for all $x \in E$ and all $n$. Prove that if $f$ is continuous then $f_n \to f$ uniformly on $E$ and show by example that convergence may not be uniform when $f$ is not continuous.

6. Let $(f_n)$ be an equicontinuous sequence of real-valued functions on $[a, b]$. Prove that if $f_n \to f$ pointwise then $f_n \to f$ uniformly.

7. Let $C[0,1]$ be the space of all continuous real-valued functions on $[0,1]$ with the metric $d$ defined by $d(f,g) = \max\{|f(x) - g(x)| : x \in [0,1]\}$. Prove that the closed unit ball $B = \{f \in C[0,1] : d(f,\phi) \leq 1\}$ is not compact, where $\phi(x) = 0$ for all $x \in [0,1]$. (Hint: consider $f_n(x) = x^n$.)

8. Let $f$ be a Lebesgue-integrable function on $[a, b]$ and define $F(x) = \int_a^x f(t)dt$ for $a \leq x \leq b$. Prove that $F$ is continuous on $[a, b]$. 

1
9. Let $C$ be a closed subset of $[a, b]$. Prove that there exists a sequence $(g_n)$ of continuous functions on $[a, b]$ such that $\lim_{n \to \infty} \int_a^b |g_n(x) - K_C(x)|^2 \, dx = 0$ where $K_C$ is the characteristic function of $C$.

10. Let $f$ be a nonnegative measurable function and $E$ a measurable set. Prove that if $\int_E f \, d\mu = 0$ then $f = 0$ almost everywhere on $E$. 