First-Year Analysis Examination  
September 2004

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Recall that the real numbers $\mathbb{R}$ form an ordered field that is complete in the sense that it enjoys the least upper bound property.  
   (i) Show that $\mathbb{R}$ has the Archimedean property: if $a, b \in \mathbb{R}$ and $a > 0$ then there exists an integer $n > 0$ such that $na > b$.  
   (ii) Show by example that an ordered field having the Archimedean property may lack the least upper bound property.

2.  
   (i) Let $(a_n : n \in \mathbb{N})$ be a real sequence converging to the real number $a$. 
   Prove that if $\pi$ is a permutation of $\mathbb{N}$ then $a_{\pi(n)} \to a$ as $n \to \infty$. 
   (ii) Show by example that rearrangements of a convergent real series need not converge.

3. Let $f : (a, b) \to \mathbb{R}$ be continuous, bounded and monotonic. Prove carefully that $f$ is uniformly continuous. 
   Suggestion: Consider $f(a+)$.  

4. Let $M$ be a compact metric space. Prove that each sequence in $M$ has a convergent subsequence. Hence, or otherwise, prove that $M$ is complete.

5. Let $f : (a, b) \to \mathbb{R}$ be differentiable. Prove or give a counterexample.  
   (i) If $f'$ is bounded, then $f$ is uniformly continuous.  
   (ii) If $f$ is uniformly continuous, then $f'$ is bounded.

6. Suppose $K : [0, 1] \times [0, 1] \to \mathbb{R}$ is a bounded function, measurable in each variable separately, such that there is a constant $C$ so that if $0 \leq x, y \leq 1$, then 
   \[ \int_0^1 |K(x, t) - K(y, t)| dt \leq C|x - y|. \]
   Suppose $(f_n : n \in \mathbb{N})$ is a uniformly bounded sequence of Lebesgue integrable functions from $[0, 1]$ to $\mathbb{R}$. Define $F_n : [0, 1] \to \mathbb{R}$ by 
   \[ F_n(x) = \int_0^1 K(x, t)f_n(t)dt. \]
   Does the sequence $(F_n : n \in \mathbb{N})$ have a uniformly convergent subsequence?
7. Fix \( a < c < b \). Show, if \( \alpha : [a, b] \to \mathbb{R} \) is increasing and continuous at \( c \) and if \( f : [a, b] \to \mathbb{R} \) is bounded and continuous except possibly at \( c \), then \( f \) is Riemann integrable with respect to \( \alpha \).

Give an example which shows that the continuity of \( \alpha \) at \( c \) is necessary.

8. Let \((X, \Sigma, \mu)\) denote a measure space. Show, if \( f \) is integrable with respect to \( \mu \), then for every \( \epsilon > 0 \) there is a \( \delta > 0 \) so that if \( A \in \Sigma \) and \( \mu(A) < \delta \), then

\[
\left| \int_A f \, d\mu \right| < \epsilon.
\]

9. Suppose \( g_n : [0, 2\pi] \to \mathbb{C} \) is a uniformly bounded sequence of Lebesgue integrable functions which converges pointwise. Let

\[
f_n(z) = \frac{1}{2i\pi} \int_0^{2\pi} \frac{g_n(t)}{\exp(it) - z} dt, \quad |z| < 1.
\]

Does the sequence \( f_n \) converges pointwise for \( |z| < 1 \)?

10. Give examples of the following, if possible. You need not prove your answer, but do give a brief justification.

(i) A bounded function \( f : [0, 1] \to \mathbb{R} \) which is not Riemann integrable.

(ii) An \( f \in C(\mathbb{T}) \) which is not in the uniform closure of the set

\[
\{ p(\gamma) = \sum_{j=0}^n p_j \gamma^j : n \in \mathbb{N}, p_j \in \mathbb{C} \}.
\]

Here \( \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \) is the unit circle, \( C(\mathbb{T}) \) denotes the continuous complex-valued functions on \( \mathbb{T} \), and \( \gamma \in \mathbb{T} \).

(iii) A differentiable function \( f : \mathbb{R} \to \mathbb{R} \) such that \( f'(0) > 0 \), but which is not increasing on any open interval containing 0.

(iv) A sequence of functions which converges in \( L^1([0, 1]) \), but not in \( L^2([0, 1]) \).