First-Year Analysis Examination
January 2004

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Let \( f : [a, b] \to \mathbb{R} \) be continuous and let \( f(a) < 0 < f(b) \). By considering the set
\[
\{ t \in [a, b] : f(t) < 0 \}
\]
or otherwise, prove carefully that there exists \( p \in (a, b) \) such that \( f(p) = 0 \).

Warning: The Intermediate Value Theorem may not be used unless it is proved.

2. Let \( f : X \to Y \) be a map between metric spaces and let \( (x_n : n \geq 0) \) be a Cauchy sequence in \( X \).
   (i) Show that if \( f \) is uniformly continuous then \( (f(x_n) : n \geq 0) \) is a Cauchy sequence in \( Y \).
   (ii) Show by example that \( (f(x_n) : n \geq 0) \) need not be Cauchy if \( f \) is merely continuous.

3. Let \( (a_n : n \geq 0) \) be a sequence of strictly positive real numbers. Show that if the series \( \sum_{n \geq 0} a_n \) converges then the series \( \sum_{n \geq 0} \sqrt{a_n a_{n+1}} \) converges. Show also that the converse is true provided the sequence \( (a_n : n \geq 0) \) is monotonic.

4. Let \( A \) be a given non-compact subset of \( \mathbb{R} \). Prove that there exist:
   (i) a continuous function \( f : A \to \mathbb{R} \) that is not bounded;
   (ii) a continuous function \( g : A \to \mathbb{R} \) that is bounded but does not attain its bounds.

5. Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function satisfying the initial value problem (IVP)
\[ f' = f \text{ and } f(0) = 1. \]
   (i) Show that if \( s, t \in \mathbb{R} \) then \( f(s - t)f(t) = f(s) \).
   (ii) Deduce that \( f \) is everywhere strictly positive.
   (iii) Conclude that the given IVP has at most one solution.

Warning: The exponential function may not be assumed.
6. Give examples of the following, if possible.
   (i) A measure space \((X, \Sigma, \mu)\) and sequence of integrable functions \(\{f_n\}\) which converges to an integrable function \(f\) uniformly, but for which the sequence \(\{\int_X f_n \, d\mu\}\) does not converge to \(\int_X f \, d\mu\).
   (ii) A bounded function \(f : [0, 1] \to \mathbb{R}\) which is not Riemann integrable.
   (iii) A closed bounded subset of \(C([0, 1])\) which is not compact.

7. Let \((X, \Sigma, \mu)\) be a measure space. Let \(\{g_n\}\) be a sequence of integrable functions which converges almost everywhere to an integrable function \(g\). Let \(\{f_n\}\) be a sequence of measurable functions such that

   \[|f_n| \leq g_n\]

and so that \(\{f_n\}\) converges to a function \(f\) almost everywhere. Show, if

   \[
   \lim \int g_n \, d\mu = \int g \, d\mu,
   \]

then

   \[
   \lim \int f_n \, d\mu = \int f \, d\mu.
   \]

**Suggestion:** Consider the proof of the dominated convergence theorem.

8. Suppose \(K : [0, 1] \times [0, 1] \to \mathbb{R}\) is continuous. Show, if \(\{f_n\}_{n=1}^\infty\) is a uniformly bounded sequence of (Lebesgue) measurable functions on the interval \([0, 1]\), then the sequence \(\{F_n\}_{n=1}^\infty\) defined by

   \[F_n(x) = \int_0^1 K(x, t) f_n(t) \, dt\]

is equicontinuous on \([0, 1]\). Must \(\{F_n\}\) have a uniformly convergent subsequence?

9. Define \(\alpha : [-1, 1] \to \mathbb{R}\) by \(\alpha(x) = x\) for \(-1 \leq x \leq 0\) and \(\alpha(x) = 1 + x\) for \(0 < x \leq 1\). Explain why \(f : [-1, 1] \to \mathbb{R}\) given by \(f(x) = \exp(x)\) is Riemann-Stieltjes integrable on \([-1, 1]\) with respect to \(\alpha\) and compute the integral.

10. Let \(\mathcal{R}\) be a \(\sigma\)-ring on a set \(X\) which is not a \(\sigma\)-algebra (so \(X \notin \mathcal{R}\)). Set \(\mathcal{R}' = \{X \setminus E : E \in \mathcal{R}\}\) and show \(\mathcal{B} = \mathcal{R} \cup \mathcal{R}'\) is the smallest \(\sigma\)-algebra containing \(\mathcal{R}\); i.e., \(\mathcal{B}\) is the \(\sigma\)-algebra generated by the collection \(\mathcal{R}\).