Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Prove that if \((a_n : n \geq 0)\) and \((b_n : n \geq 0)\) are bounded sequences of real numbers then

\[
\liminf_{n \to \infty} (a_n + b_n) \geq \liminf_{n \to \infty} (a_n) + \liminf_{n \to \infty} (b_n).
\]

Show by example that the inequality can be strict.

2. Let \((a_n : n \geq 0)\) be a sequence of positive real numbers and assume that the series \(\sum_{n=0}^{\infty} a_n\) diverges.
   (a) Does the series \(\sum_{n=0}^{\infty} \frac{a_n}{1+a_n}\) converge or diverge? Explain.
   (b) Does the series \(\sum_{n=0}^{\infty} \frac{1}{1+n^{2}a_n}\) converge or diverge? Explain.

3. Let \(M\) be a metric space and \(f : M \to \mathbb{R}\) a continuous function. Show that if \(M\) is compact then \(f\) is necessarily bounded. Does boundedness of \(f\) follow from precompactness of \(M\)? Explain.
   [Recall that by definition, \(M\) is precompact when for each \(\epsilon > 0\) there exists a cover of \(M\) by finitely many \(\epsilon\)-balls.]

   Let \(n\) be a positive integer and let \(f : [0, 1] \to \mathbb{R}\) be a continuous function with \(f(0) = f(1)\). Show that there exist points \(a, b \in [0, 1]\) such that \(b-a = \frac{1}{n}\) and \(f(a) = f(b)\).
   [It might help to consider \(g(t) = f(t) - f(t - \frac{1}{n})\) for \(t\) in a suitable interval, along with the particular sum \(g(\frac{1}{n}) + g(\frac{2}{n}) + \cdots + g(1)\).]

5. State the Mean Value Theorem.
   Let \(f : (a, b) \to \mathbb{R}\) be continuous everywhere and differentiable except perhaps at \(p \in (a, b)\). Show that if \(\lim_{t \to p} f'(t)\) exists and equals the real number \(L\), then \(f\) is differentiable at \(p\) and \(f'(p) = L\). Is it necessary to assume that \(f\) be continuous everywhere? Explain.

6. Let \(F : [a, b] \to \mathbb{R}\). Show, if \(F\) is differentiable on \([a, b]\) and \(F'\) is Riemann integrable on \([a, b]\), then

\[
F(b) - F(a) = \int_{a}^{b} F'(x) dx.
\]
7. Suppose \( \{f_n\} \) is a uniformly bounded sequence of Lebesgue integrable functions on the interval \([0, 1]\). Let

\[
F_n(x) = \int_0^x f_n(t) dt.
\]

Does the sequence \( \{F_n\} \) have a uniformly convergent subsequence?

8. Let \( \mathcal{M} \) denote the Lebesgue measurable sets in the real line \( \mathbb{R} \) and suppose \( \mu : \mathcal{M} \to \mathbb{R} \) is a (countably additive) measure. Show that if \( \mu \) is regular and \( \mu((\neg \infty, t)) = 0 \) for all \( t \in \mathbb{R} \) then \( \mu = 0 \).

[Recall that \( \mu \) \textit{regular} means that for each \( A \in \mathcal{M} \) there exist Borel sets \( F \) and \( G \) with \( F \subset A \subset G \) and \( \mu(G \setminus A) = 0 = \mu(A \setminus F) \).]

9. Give (with brief justification) examples of the following.
   (a) A sequence of continuous functions \( f_n : [0, 1] \to \mathbb{R} \) which converges pointwise to a continuous function \( f : [0, 1] \to \mathbb{R} \), but where the convergence is not uniform.
   (b) A bounded function which is not Riemann integrable.
   (c) A ring \( \mathcal{R} \) and a countably additive set function \( \phi : \mathcal{R} \to [0, \infty) \) which cannot be extended to a countably additive set function on a \( \sigma \)-ring that contains \( \mathcal{R} \).

10. Let \((X, \Sigma, \mu)\) denote a measure space. Suppose \( F : (0, 1) \times X \to \mathbb{R} \) satisfies each of the following conditions:
    (a) for each \( t \in (0, 1) \) the function \( f_t : X \to \mathbb{R} \) given by \( f_t(x) = F(t, x) \) is measurable;
    (b) for each \( x \in X \) the function \( g_x : (0, 1) \to \mathbb{R} \) given by \( g_x(t) = F(t, x) \) is continuous;
    (c) there exists a \( \mu \)-integrable function \( G : X \to [0, \infty) \) with the property that \( |F(t, x)| \leq G(x) \) for all \( t \) and \( x \).

Show that

\[
\Phi(t) = \int_X F(t, x) d\mu(x)
\]

is defined and continuous on \((0, 1)\).