First-Year Analysis Examination
January 2003

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Suppose $X$ and $Y$ are metric spaces. Prove, if $X$ is compact and $f : X \to Y$ is continuous, then $f(X)$ is compact.

2. State a theorem which generalizes the alternating series test and use it to determine those $z \in \mathbb{C}$ for which the series
   \[ \sum_{n=2}^{\infty} \frac{z^n}{\sqrt{n}} \]
   converges.

3. Give an example of a function which is continuous, but not uniformly continuous. Prove, if $X$ and $Y$ are metric spaces, $f : X \to Y$ is uniformly continuous, and $\{x_n\}$ is a Cauchy sequence in $X$, then $\{f(x_n)\}$ is a Cauchy sequence in $Y$.

4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous at every point of $\mathbb{R}$ and is differentiable except possibly at 0. If
   \[ \lim_{x \to 0} f'(x) = L \]
   (that is the limit exists as a finite number $L$), does it follow that $f$ is differentiable at 0?

5. Define $\alpha : [-1, 1] \to \mathbb{R}$ by $\alpha(x) = 0$ if $x \leq 0$ and $\alpha(x) = 1$ if $x > 0$. Show, a bounded function $f : [-1, 1] \to \mathbb{R}$ is Riemann integrable with respect to $\alpha$ if and only if $f(0+) = f(0)$.

6. Suppose $K : [0, 1] \times [0, 1] \to \mathbb{R}$ is continuous. Show, if $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded sequence of (Lebesgue) measurable functions on the interval $[0, 1]$, then the sequence $\{F_n\}_{n=1}^{\infty}$ defined by
   \[ F_n(x) = \int_{0}^{1} K(x, t)f_n(t)dt \]
is equicontinuous on $[0,1]$. Must $\{F_n\}$ have a uniformly convergent subsequence?

7. Let $L^2$ denote $L^2(\mu)$, where $\mu$ is Lebesgue measure on $[-\pi, \pi]$. Suppose $\{\phi_n\}_{n=0}^\infty$ is an orthonormal sequence from $L^2$ and there is a $C$ such that $|\phi_n(x)| \leq C$ for all $n \in \mathbb{N}$ and $x \in [-\pi, \pi]$. Show $\{\phi_n\}$ does not converge pointwise (in the interval $[-\pi, \pi]$).

Prove that the sequence of functions $f_n : \mathbb{R} \rightarrow \mathbb{R}$, $f_n(x) = \sin(nx)$ does not converge pointwise.

8. Let $(X, \Sigma, \mu)$ be a measure space. Show, if $\mu(X) < \infty$ and if $\{f_n\}$ is a sequence of integrable functions which converges uniformly to a function $f$, then $f$ is integrable and

$$\lim_{n \to \infty} \int_X |f_n - f| d\mu = 0.$$ 

Does the conclusion hold if the hypothesis $\mu(X) < \infty$ is omitted?

9. State Fatou’s Lemma and use it to prove the Dominated Convergence Theorem.